Fukaya cats. sheaves cosheaves. 2/9/2015 L

Basics. <u>Defn</u>. Let M be a C^∞ mfld of dim $2n$. $n \geq 0$. A <u>symplectic form</u> on M is a choice

$$\omega \in \Omega^2_{dR}(M;\mathbb{R})$$

s.t. · $d\omega = 0$.

· $\omega^{\wedge n}$ defines a volume form.

A pair (M, ω) is called a symplectic mfld.

<u>Rmk</u>. ω defines an isomorphism

$$\Gamma(TM) \longrightarrow \Gamma(T^*M)$$
$$v \longmapsto \omega(v, -)$$

Given H. $\rightsquigarrow dH \overset{\omega}{\rightsquigarrow} X_H$.

Example: · $M = \emptyset$, pt.

· $M = \mathbb{R}^2$. $\omega = dx \wedge dy$ or $\omega = dy \wedge dx$.

· $M = \mathbb{R}^{2n}$ $\omega = \sum_{i=1}^n dx_i \wedge dy_i$

· Any orientable 2-mfld. (with a volume form)

· For any C^∞ mfld Q, $M = T^*Q$.

On T^*Q, $\exists$ a canonical 1-form, called the Liouville form θ, which is defined as:

$$\theta(v)_{(q,p)\in T^*Q} = p(d\pi(v)).$$

Here, $v \in T(T^*Q)$, $T^*Q \xrightarrow{\pi} Q$ is the projection.

Exercise : $d\theta$ defines a symplectic form on ~~T~~ T^*Q.

- If M is a Kähler manifold, $h = g + i\omega$, the imaginary part of the Hermittian metric is symplectic.

Defn. A submanifold $L \subset M$ is called Lagrangian if

- $\dim L = \frac{1}{2} \dim M$
- $\omega|_L = 0$.

Why are Lagrangians important?

Thm (Gromov). In the standard symplectic structure $(\mathbb{R}^{2n}, \omega = \sum dx_i \wedge \ldots$ any cpt Lagrangian has $H^1_{dR}(L; \mathbb{R}) \neq 0$.

- $\exists$ symplect structures on $\mathbb{R}^{2n}$ s.t. $S^n \subset \mathbb{R}^{2n}$ is a Lagrangian.

Example $\dim M = 2$. Then any curve $L \subset M$ is lag.

Example. $L_1 \subset M_1$, $L_2 \subset M_2$ Lagrangians.

$L_1 \times L_2 \subset (M_1 \times M_2, \omega_1 \oplus \omega_2)$ is a Lagrangian.

Definition. An endomorphism $J : TM \circlearrowleft$ is called an almost complex structure if $J^2 = -1$.

J is called compatible with ω if $\omega(-, J-)$ defines a Riemannian metric.

<u>Defn</u>. A C^∞ map $u : (M_0, J_0) \to (M_1, J_1)$ is called (J_0, J_1)-holomorphic if $du \circ J_0 = J_1 \circ du$.

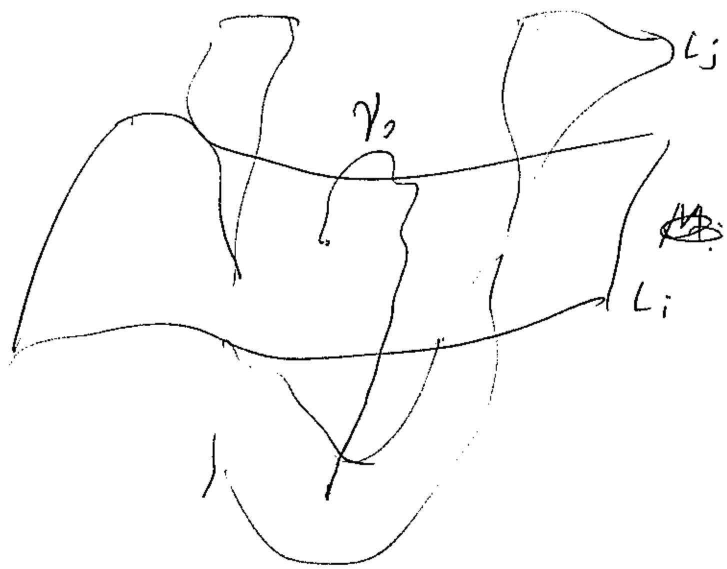
Example. If (M_i, J_i) are cplx mflds, then $u : M_0 \to M_1$ is holomorphic $\Leftrightarrow$ $du \circ J_0 = J_1 \circ du$.

<u>Toward Fukaya categories</u>

Let's fix some Lagrangians $L_0, L_1, \cdots \subset M$

Assume $L_i \pitchfork L_j$

Do Morse theory on $P(L_i, L_j) = \{ \gamma : [0,1] \to M \mid \gamma(0) \in L_i, \gamma(1) \in L_j \}$



Instead of constructing a function on $P(L_i, L_j)$, we specify the derivative of that function.

Fix some component $P_0 \subset P$. Fix a base point $\gamma_0 \in P_0$. Then $\forall \gamma \in P_0$, $\exists$ some path from γ_0 to γ.

i.e. $u : [0,1] \times [0,1] \to M$

s.t. $u(0,-) = \gamma_0$. $u(1,-) = \gamma$.

take $\mathcal{A}(\gamma) = \int_{[0,1]\times[0,1]} u^*\omega$.

Then the rate of change of $\mathcal{A}(\gamma)$ can be well-defined.

We do Morse theory with $d\mathcal{A}$.

We need a metric to make $d\mathcal{A}$ a vector field.

Suffices to fix a "metric" on M, we take $g = \omega(-, J-)$

Lemma. $d\mathcal{A}(\gamma) = 0 \Leftrightarrow \gamma$ is a constant map.

(i.e. $\{\gamma \mid d\mathcal{A}(\gamma) = 0\} \cong L_i \cap L_j$)

The gradient trajectories are maps

$$u : \cancel{[0,1]}\ \mathbb{R} \times [0,1] \to M$$

s.t. (i) $u|_{\mathbb{R}\times\{0\}} \subset L_i$

$u|_{\mathbb{R}\times\{1\}} \subset L_j$

$\lim_{\cdot \to +\infty} u(\cdot, s) = p \in L_i \cap L_j$

$\lim_{\cdot \to -\infty} u(\cdot, s) = q \in L_i \cap L_j$

(ii) u is $\cancel{(J,j)}$ (j, J)-holomorphic

where j is the standard cplx structure on $\mathbb{R}\times[0,1]$.

To do Morse theory, we need

- to "count" certain trajectories
  - ↑ 0-dim families
- to know $d^2 = 1$.

  {trajectories} has a 1-dim component where boundary recover the 0-dim component.