Math 277: Fukaya categories, Fall 2015

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1. 2015-9-1

Based on the two people who have filled out the survey so far (fill it out!), it seems we are collectively uncomfortable with some basic objects that are relevant. So we will stick with the basics for the first half of today.

The first half of this course will give three good easy examples of mirror symmetry.

Let's start the math.

1.1. **Basics.**

Let M be a C^∞ manifold of dimension $2n$. A *symplectic form* on M is a choice of closed form $\omega \in \Omega^2_{\mathrm{dR}}(M, \mathbb{R})$ such that ω^n defines a volume form. A pair (M, ω) is called a *symplectic manifold.*

Remark. ω defines an isomorphism $\omega : \Gamma(TM) \to \Gamma(T^*M)$ given by applying ω to the tangent vector. This has an analogue in physics, where given a function H on a manifold, we want to compute alternately dH or its associated vector field X_H.[1]

Examples of symplectic manifolds include points, arguably the empty set depending on what we consider its dimension to be, and also nonsilly things. $\mathbb{R}^2$ with whatever nonzero constant choice of 2-form works, for instance. $M = \mathbb{R}^2 n$ has 2-forms $\sum_i dx_i \wedge dy_i$. Any orientable 2-manifold works for obvious reasons. Perhaps most interestingly, we have, for any smooth manifold Q, that the cotangent bundle T^*Q—on which we have the canonical 1-form (the Liouville form) given by $\theta(v)_{(q,p)\in T^*Q} = p(d\pi(v))$—has the symplectic form $d\theta$.

Exercise 1.1. Verify that $(T^*Q, d\theta)$ is a symplectic manifold.

[1] The significance of this statement in physics is unclear to Geoff; he apologizes for the lack of context

A final example is a Kähler manifold, for which the imaginary part of the hermitian metric is symplectic.

Now, call a submanifold $L \subset M$ of the symplectic manifold M a *Lagrangian* if $\dim L = \frac{1}{2} \dim M$ and $\omega|_L = 0$. This object seems artificially defined to us at first glance, but we have the following nice theorem indicating a potential for deeper significance.

Theorem 1.2 (Gromov)**.** *We have the following.*

- *In $\mathbb{R}^{2n}$ with a usual symplectic structure, any compact Lagrangian has $H^1_{\mathrm{dR}}(L, \mathbb{R}) \neq 0$.*
- *There exist symplectic structures on $\mathbb{R}^{2n}$ such that $S^n \subset \mathbb{R}^2 n$ is a Lagrangian.*

An *almost complex structure* on a manifold M is a endomorphism J of TM such that $J^2 = -1$. J is called *compatible* with ω if $\omega(\bullet, J\bullet)$ is a Riemannian metric. A smooth map $u : (M, J) \to (M', J')$ is called (J, J')*-holomorphic* if $du \circ J = J' \circ du$. For instance, if these are complex manifolds, the J, J'-holomorphic maps are precisely the holomorphic maps.

1.2. **Toward Fukaya categories.** This will be nonrigorous! Feel the demand for rigor churn in your stomach, and deal with it somewhere else.

Fix Lagrangians $L_0, \dots$ of M, and assume L_i, L_j intersect simply transversally. We play a fun game. This game is called Morse theory. Set $P(L_i, L_j) = \{\gamma : [0, 1] \to M | \gamma(0) \in L_i, \gamma(1) \in L_j\}$.

Fix some component $P_0 \subseteq P$, fix a basepoint $\gamma_0 \in P_0$. Then for all paths $\gamma \in P_0$, we have some path from γ_0 to γ, that is, a function $u : [0, 1] \times [0, 1] \to M$, so we can take the number $A(\gamma) = \int_{[0,1]\times[0,1]} u^*\omega$. Then the *rate of change* of A is independent of γ_0 and of u itself (ω is closed!). We do Morse theory with dA. This will be our basic function. Because Morse theory needs it apparently, we also need a metric on P_0, but we can abduct it from a metric on M by integrating the length of the perturbation of the path over the path. So now we need a metric on M, which we get by choosing an almost complex structure J and setting $g = \omega(\bullet, J\bullet)$.

The following lemma is more difficult than it looks because infinite dimensional analysis is not easy.

Lemma 1.3 (Floer)**.** *$dA(\gamma) = 0$ iff γ is a constant path. So the set of critical points of A are in bijection with points $L_i \cap L_j$.*

The *gradient trajectories* are maps $u : \mathbb{R} \times [0, 1] \to M$ such that $u|_{\mathbb{R}\times\{0\}} \subset L_i$, $u|_{\mathbb{R}\times\{1\}} \subset L_j$, $\lim_{\to\infty} u(\bullet, s) = p \in L_i \cap L_j$, $\lim_{\to-\infty} u(\bullet, s) =$

$q \in L_i \cap L_j$, and u is (i,J)-holomorphic treating $\mathbb{R} \times [0,1]$ as a subset of $\mathbb{C}$.

Geoff left the lecture here, with a few minutes left in the class. Apparently there will be other notes, and he recommends looking at those.