LECTURE 1

Toward the Fukaya category

Some remarks: (I.e., things I said out loud that did not make it to the board)

(1) I'll start basic; the facts I'll assert about Fukaya categories probably don't seem so impressive or appreciable until we have some basic facts and examples under our belts. So before asserting such facts, I want to set up the basic ideas. The goal is to start playing with games in a symplectic manifold, and asking what kinds of structures it has algebraically. We'll start seeing an algebraic structure, and the name we'll give to this structure is that of an A_∞ category.

(2) The condition that $d\omega = 0$ is the analogue of the integrability condition for holomorphic manifolds. The rigidity phenomena that you're familiar with from complex geometry have analogues in the symplectic world, and they are mostly due to this closedness condition. As a dumb example, when you want to compute the Lie derivative of a symplectic form, you'd use the Cartan formula. Then you see that one of the terms goes away immediately.

(3) Symplectic geometry really comes from classical mechanics. There, you learn that a single Hamiltonian called H, which encodes energy, completely determines your dynamical system. (I.e., how your states evolve.) Well, if you imagine that M is your phase space, then any H determines a vector field on M: Just take its derivative, then use the symplectic form to turn it into a vector field. The flow of this vector field is precisely the evolution of the dynamical system encoded by H.

(4) The J being compatible with ω is another instance of the 2-out-of-3 idea you've probably seen in Kahler geometry. Of Riemannian, symplectic, and complex structures, any two determine the third. The reason that compatibility is so useful for us is that, even if J is not integrable (hence even though J does not define a holomorphic structure) we can still hope for some sort of rigidity in the analysis of holomorphic maps into M. This compatibility with ω, and the rigidity from $d\omega = 0$, is responsible for the rigidity you'll see.

(5) There are serious analytical difficulties and subtleties in even talking about proper notions of tangent spaces to infinite-dimensional manifolds. So all the ideas and results in this lectures are things you can do using back of the napkin computations, but things you might not know how to formally define as a gradient flow on an infinite-dimensional manifold, or as tangent spaces to an infinite-dimensional manifold. But I promised that even without setting up such formalities for rigorous proof, you could go home and discover this Lemma on your own. So you'd know what you'd want your formal set-up to recover, but you will have to do work to make your napkin computations into real papers. An example is that the tangent space to $\gamma \in \mathcal{P}$ is given by the vector space of tangent vector fields to γ. That is, a tangent vector to γ is just a vector field along γ; heuristically, this is the thing that tells you how to "nudge" or "deform" γ. Given two such vector fields, we know how to get a number out of them if we have a Riemannian metric on M itself: You take the inner product at every point of γ, then integrate the resulting function. That is, the Riemannian metric on $\mathcal{P}$ is defined by

$$\langle X, Y \rangle = \int \langle X(\gamma(t)), Y(\gamma(t)) \rangle dt. \tag{0.1}$$

(6) If $V \subset \mathbb{C}$ is open and $u : V \to (M, \omega, J)$ is holomorphic, what does $\int_V u^*\omega = 0$ imply?