LECTURE 2

Morse Theory

(1) Given any critical point of f, you can look at its ascending manifold. This is the set of all points x that are obtained from the critical point via a gradient trajectory. The descending manifold, in contrast, is the set of all points that reach the critical point via a gradient trajectory. The two intersect transversely at the critical point itself.

(2) Given any two critical points, you can ask whether their ascending descending manifolds intersect. If they intersect transversally, and you orient them both, the intersection defines an oriented submanifold of X. If this oriented submanifold is one-dimensional, it represents a single, discrete Morse trajectory. These are what we count to define differentials.

(3) This data also allows you, eventually, to realize X as a CW complex. This recovers the homotopy type of X if you can analyze the attaching maps.

(4) An even simpler thing to do is just count the trajectories with sign, which gives a chain complex whose homology recovers the homology of X.

(5) You should read Bott's Morse Theory indomitable. In it, you'll see what goes wrong with the torus example. That highly symmetric Morse function is not generic enough.

(6) Do Morse theory on a compact smooth manifold using f, then using $-f$. What does this say?

(7) How is the Morse complex of (X, f) and (Y, h) related to the Morse complex of $(X \times Y, f + h)$?