Hiro (2)
9/4/2015

Exercise Let $\mathcal{U} \overset{\text{open}}{\subset} \mathbb{C}$ be open, and $u : \mathcal{U} \to (M, \omega, J)$ be holomorphic.

(symplectic mfd.; J: alm. cpx str. compatible w/ ω)

What does $\int_{\mathcal{U}} u^*\omega = 0$ imply?

($du \circ j_{\mathcal{U}} = J \circ du$.)

Morse theory Fix X C^∞-mfd, g Riemannian metric, $f : X \to \mathbb{R}$ C^∞-func.

"generic f will tell you a lot about the space X!"

One can define a graded abelian group out of this data (X, f).

Using g, we'll make a differential. In good cases, $H^*(\text{this chain complex}) \simeq H^*(X)$

Can recover homotopy type of X but cannot diffeom type of X.

This gr. abel. grp has generators $\mathrm{Crit}(f) = \{x \in X \text{ s.t. } df_x = 0\}$

To have, for instance, a discrete set of generators, we'll ask that f be "sufficiently general" (e.g. f shouldn't be constant).

Lem $\forall$ C^∞ mfd X, almost every C^∞ func f satisfies the "Morse" property:

$\forall x \in \mathrm{Crit}(f)$, $\exists$ coordinate chart about x s.t. $f = \sum_{i=1}^{k} x_i^2 - \sum_{k+1}^{\dim X} x_i^2$.

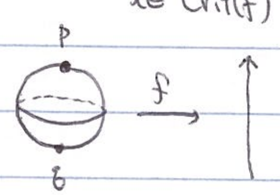
(proof: use Taylor exp. & fund. thm of calculus).

Def. # of negative signs, $\dim X - k$, is called the index of f at x, $\mathrm{ind}(x)$.

The gr. abel grp is $\bigoplus_{x \in \mathrm{Crit}(f)} \mathbb{Z}[\mathrm{ind}(x)]$

(possibly $\pm$ (depending on convention: hom vs cohom))

e.g. $X = S^2$



(height func in $S^2 \subset \mathbb{R}^3$) Near p, $f = -x_1^2 - x_2^2$ $\mathrm{ind}(p) = 2$

q $f = x_1^2 + x_2^2$ $\mathrm{ind}(q) = 0$

$$\begin{array}{cc} 2 & \mathbb{Z} \\ 1 & 0 \\ 0 & \mathbb{Z} \end{array} \quad \left(\text{or} \quad \begin{array}{cc} 0 & \mathbb{Z} \\ -1 & 0 \\ -2 & \mathbb{Z} \end{array} \right)$$

$\partial^2 = 0$

Define a (putative) differential : $\partial p = \sum_{\substack{|q| = |p| - 1 \\ \text{ind}(p)}} n(p,q)\, q$.

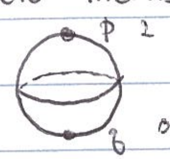
where $n(p,q) = \#\{$discrete gradient trajectories from q to $p\}$ ← different from the usual convention. $\{$from p to q ; $\dot{c} = -\nabla f$.

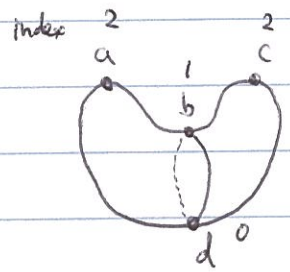
(A gradient trajectory is a C^∞ map $c : \mathbb{R} \to X$ s.t. $\dot{c}(t) = \nabla f(c(t))$.)

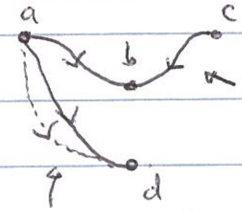
use metric g: $df \longleftrightarrow \nabla f$

$\Omega^1 \overset{g}{\cong} \Gamma(TM)$

"discrete" means :

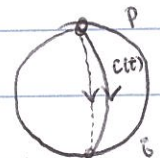






← moduli space is "0-dim" ("discrete")

moduli space is "continuos"



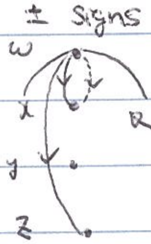
Here the (moduli) space of trajectories from p to q (mod translation) is S^1 ← continuous ← cf. no index 1 pt in S^1 → ∂ should be 0 anyway.

Rem following usual convention.

→ $n(p,q) = \#\{$ points in a dimension zero moduli space $\}$

← count with $\pm$ signs (coming from (local) orientation)

e.g.



$\partial w = x - x + \underbrace{y - y}_{??\ (\text{may not exist?})}$.

Q. When is ∂ a differential?

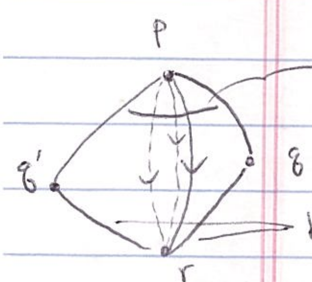
A. Given our current assumption, not always.

Why? proof (sketch of $\partial^2=0$) The idea is that if I is some compact 1-mfd, possibly w/ ∂, then ∂I occurs in pairs.

If I is oriented, the signed count of $\partial I = 0$.

So let's realize the "broken trajectories" as ∂ of a 1-mfd.

$$\partial^2 p = \sum_{|q|=|p|-1} \sum_{|r|=|p|-2} \#\{\text{traj. } r\to q\}\cdot\#\{\text{traj } q\to p\}\, r$$

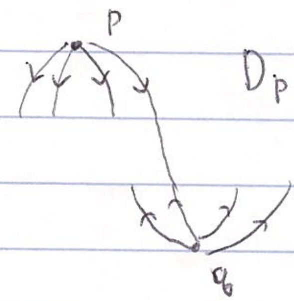


1-dim freedom

forms "an open moduli space"

add them & get "closed 1-dim moduli sp" I.

broken trajectories



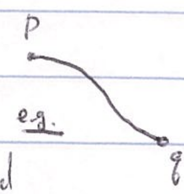
$D_p = \{x \mid \exists$ a gradient traj passing through x & ending at $p\}$ "descending" mfd.

$A_q = \{x \mid \exists$ a grad traj passing through x & beginning at $q\}$

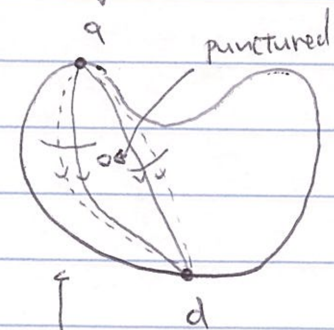
"ascending mfd"

$$\begin{cases} \text{index} \longleftrightarrow \dim D_p,\ \dim A_q. \\ D_p \cap A_q \longleftrightarrow \text{grad traj } p \to q. \end{cases}$$

e.g.



example where this argument fails:



"moduli" =

due to the punctured pt.