1 September 4.

Exercise 1.1. Choose a nonempty open set $V \subset \mathbb{C}$. Let M be a symplectic manifold with almost complex structure J compatible with ω. Let $u : V \to M$ be a holomorphic map. What does it mean if $\int_V u^*\omega = 0$?

1.1 Morse Theory introduction

Fix X a C^∞ manifold, g a Riemannian metric, $f : X \to \mathbb{R}$ a C^∞ function. One can define a graded abelian group from the data (X, f). Using g (in good cases), we can construct a differential. Then (in good cases) $H^\bullet$ of this chain complex is the same as $H^\bullet(X)$.

How do we construct this group? Its generators are

$$\mathrm{Crit}(f) = \{x \in X \mid df_x = 0\}.$$

To have a discrete set of generators, we need f to be "sufficiently generic".

Lemma 1.1. *For all smooth manifolds X, almost every C^∞ function f satisfies the **Morse property**, which means that for all $x \in \mathrm{Crit}(f)$, there is a coordinate chart around x such that*

$$f = \sum_{i=1}^{k} x_i^2 - \sum_{i=k+1}^{\dim X} x_i^2.$$

Definition 1.1. The number of negative signs (that is, $\dim X - k$) is called the **index** of f at x, and written $\mathrm{Ind}(x)$.

The graded abelian group we want is then

$$\oplus_{x \in \mathrm{Crit}(f)} \mathbb{Z}[\mathrm{Ind}(x)]$$

(where [] means to shift $\mathbb{Z}$ to the given grade).

Example 1.1. If $X = S^2$, and f is the projection of X onto a vertical line, and p, q are the corresponding poles (p on top), then near p, we have $f = -x_1^2 - x_2^2$, so $\mathrm{Ind}(p) = 2$; near q, we have $f = x_1^2 + x_2^2$, so $\mathrm{Ind}(q) = 0$. Thus the graded abelian ring has a $\mathbb{Z}$ in degrees 0 and 2. (Depending on convention and whether we are doing homology or cohomology, we might use 0 and -2 instead.)

Write $|p|$ for the degree of the generator of the graded abelian group corresponding to p (that is, $\mathrm{Ind}(p)$). We define a (putative) differential

$$\partial p = \sum_{|q|=|p|-1} n(p, q) q$$

where $n(p, q)$ is the number of discrete gradient trajectories from q to p. Here a **gradient trajectory** is a C^∞ map $c : \mathbb{R} \to X$ such that $\dot{c}(t) = \nabla f(c(t))$ (we get ∇ from the Riemannian metric). (Note we actually want some signs on $n(p, q)$ but we won't talk about that yet.)

For example, the space of trajectories from the poles p to q on S^2 is S^1, so there are no discrete trajectories. Another example is a sort of croissant shape, curved upward: if we let a, c be the critical points on the ends of the croissant, b the point on the inside of the curve, and d the point on the outside of the curve, then a is index 2, b is index 1, and d is index 0; there is a single trajectory from a to b, and all other trajectories from a go to d.

In general, one can show that the dimension of the space of trajectories from an index k point to an index l point is of dimension $k - l - 1$. If this number is 0, we possibly have a finite number of discrete trajectories, which we can count. Another way to say this is that $n(p, q)$ is the number of points in a certain zero-dimensional moduli space.

So, when is ∂ a differential? Given our current assumptions, not always.

When it is, the idea of the proof is that if I is some compact 1-manifold, possibly with boundary, ∂I comes in pairs of points; if I is oriented, the signed count of ∂I is 0. So we can try to realize the "broken trajectories" appearing in the coefficients of ∂^2 as the boundary of a 1-manifold. That is, we have

$$\partial^2 p = \sum_{|q|=|p|-1} \sum_{|r|=|p|-2} \#\{r \to q\}\#\{q \to p\}r.$$

The space of trajectories from p to r is a one-dimensional moduli space. The "broken trajectories" counted by the coefficients in the above sum are just the boundary points of this space. This can break down when, for example, our manifold has a puncture.

Let D_p be the set of points x such that there is a trajectory passing through x and ending at p; this is the "descending manifold" from p. Similarly, let A_q be the set of x such that there is a trajectory passing through x and beginning at q; this is the "ascending manifold" from q. These are submanifolds of X which intersect in the set of trajectories we're interested in (this is why that set has dimension $\mathrm{Ind}(p) - \mathrm{Ind}(q) - 1$—you can explicitly count the dimension of the space of descending trajectories from the vertex of a hyperbola). Upon choosing orientations for D_p and A_q, we get the signs that we put on $n(p, q)$.