Hiro's Class (~~3rd~~ 4th class)

Exercises:
- Let $\dim M \neq 0$. Show if $\omega = d\theta$ for some $\theta \in \Omega^1$, M must be non-compact (or have boundary). (Such an M is called exact.)

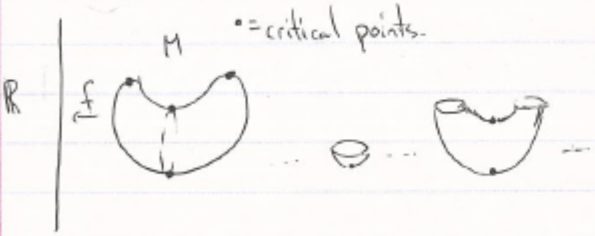
- Let $H: M \to \mathbb{R}$ be a C^∞ function, and X_H the vector field dual to $dH \in \Omega^1$. Let $\Phi^H: M \times \mathbb{R} \to M$ denote the flow. Show that if $L \subset M$ is Lagrangian, then so is

this is $dy\,dx$, or $dp\,dq$.
$$L \times \mathbb{R} \hookrightarrow (M \times T^*\mathbb{R},\ \omega_M \oplus \omega_{T^*\mathbb{R}})$$
$$(x,t) \longmapsto \left(\Phi^H(x,t),\ t, -H(\Phi^H(x,t))\right)$$

- Show H is constant along the flow of X_H.

Last time: Broad overview of Morse theory.

Given X compact, $f: X \to \mathbb{R}$, g a metric w/ f & g suff. generic, we can construct a (co) chain cx w/ generators $\longleftrightarrow \mathrm{crit}(f)$
$$d \longleftrightarrow \#\text{ of } \nabla \text{ flows.}$$
s.t. $H_\bullet^{Morse}(X, f, g) \cong H_\bullet(X;\mathbb{Z})$ (does NOT descend from a q-iso of chain cxs!)



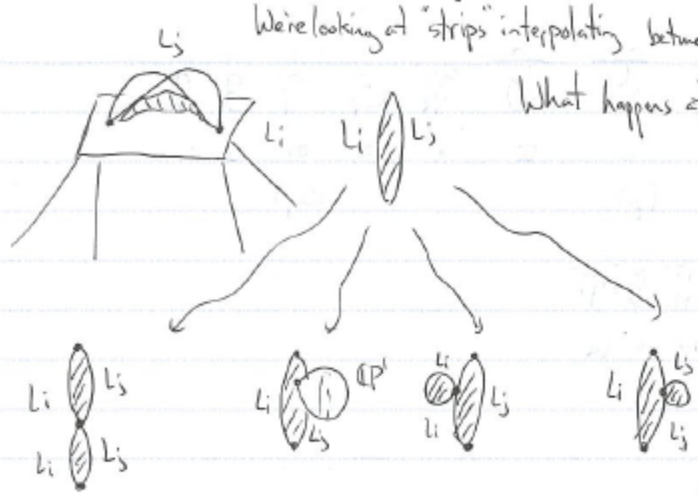
We'll go back to fantasizing about "compactifying" 1-dim'l moduli spaces of holomorphic strips (polygons)

Recall: Fix $L_1, \dots, L_n \subset M$. fixed J compatible w/ ω. Fix i, j.

$$\left\{ u: \mathbb{R}\times[0,1] \to M \text{ s.t. } \begin{array}{l} u(t,0) \subset L_i \\ u(t,1) \subset L_j \end{array} \quad \lim_{t \to \pm\infty} u(t,\cdot) \subset L_i \cap L_j \right\} \Big/ \mathbb{R}$$
$$\text{and } J \cdot du = du \cdot i_{\mathbb{C}}$$

Let's suppose we can pick out a 1-dim ($\mathbb{R}$ dim) space of such strips. What does its compactification look like?

We're looking at "strips" $\overset{u}{}$ interpolating between L_i & L_j.



What happens at the boundary of this moduli space?

Seems like doing Morse theory as usual (defining d s.t. $dp = \sum_q \#\{\text{holom strips from } q \text{ to } p\}\, q$) results in $d^2 = 0$! b/c sum of all 4 types is 0, and represents d^2.

($d^2 = \sum + \sum + \sum$)
(but if we knew these "bubbles" can't appear, then $d^2 = 0$)

So, <u>Def:</u> (M, ω) is called <u>exact</u> if $\omega = d\vartheta$ for some $\vartheta \in \Omega^1 M$.
$L \subset M$ is called <u>exact</u> if $\exists f : L \to \mathbb{R}$ s.t. $\vartheta|_L = df$
These conditions exclude the bubbles!

(Exercise from last time: $\int_{\mathbb{CP}^1} u^* \omega = 0 \Leftrightarrow u$ constant.)

$\omega = d\vartheta \Rightarrow \int_{S^2} u^*\omega = \int u^* d\vartheta \overset{\text{parts}}{=} \int du^* \vartheta \overset{\text{Stokes}}{=} \int_{\partial S^2} u^*\vartheta = 0$ rules out $\mathbb{CP}^1$ bubbles. Same trick eliminates the others. (using $\vartheta|_L = df$)
∂S^2 empty.

<u>Thm:</u> $\exists$ a setup (a perturbed version of the aforementioned) in which, if we let
$CF^*(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}[|p|]$ w/ $dp = \sum_q \#\{\text{holom strips } q \to p\}\, q$

then $d^2 = 0$ (Assuming M exact, L_i exact, spin)
(the tangent bundle comes equipped w/ a map to $BSO(n)$ (oriented). "Spin" means this lifts to a map to $BSpin$