Harvard Math 277 – Hiro Tanaka –

Kevin Sackel — Lecture 4 — 11 Sept 2015

- **Exercise**: Let M be a cpt mfld with $\partial M = \emptyset$. Show that if any of M's even dimensional H^*_{dR} vanish ($\leq \dim M$), M cannot be symplectic

- **Exercise**: Let $M = T^*Q$, Q a C^∞ mfld. Given any $Z \subset Q$ smooth submfld, define
$$T^*_Z Q := \{(z,\alpha) \mid z \in Z,\ \alpha \in T^*Q|_Z,\ \alpha|_{TZ} = 0\}$$
Show this is an exact Lagrangian submfld

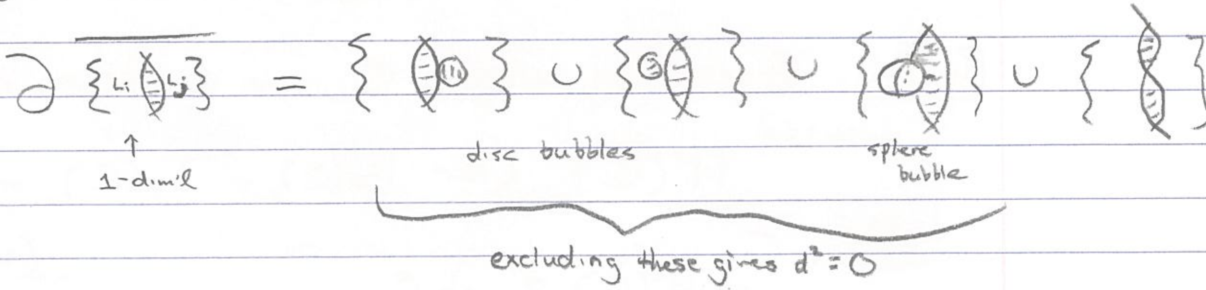
- **Exercise**: Fix (M,ω). Prove a compatible J exists. Prove the space of such J is contractible.

- **Remark**: When defining "invariants," want it not to depend on choices. Our chain complexes depend on J, but hopefully these should all be equivalent because the space of J's is contractible. This is the philosophy, though the analysis is a bit difficult.

  (Think: Morse (co)homology doesn't depend on metric or Morse function)

- **Last Time**: Motivated choice of (M,ω), $\omega = d\Theta$ (exact sympl. mfld)
$$L_i \subset M \text{ s.t. } \Theta|_{L_i} = df_i \quad f_i : L \to \mathbb{R} \quad \text{(exact Lagrangian)}$$

  Why?



Thm (no pf, no details): $\forall$ such L_i, under good conditions, can define a cochain complex
$$(CF^*(L_i, L_j), d) \rightsquigarrow \text{Floer cochain complex}$$
generated by $L_i \pitchfork L_j$, and with d counting $\#\{\text{holo}^c \text{ strips}\}$

- **Defn**: Fix a ring k. A dg category $\mathcal{C}$ over k is the data of:

  (1) A "set" of objects $\mathrm{ob}\,\mathcal{C}$

  (2) $\forall$ pairs of objects $X, Y \in \mathrm{ob}\,\mathcal{C}$, a cochain complex $\hom^{\cdot}(X,Y)$

  i.e. $\hom^{\cdot}(X,Y) := \Big(\bigoplus_{i\in\mathbb{Z}} \underbrace{\hom^{i}(X,Y)}_{k\text{-module}}, d\Big)$

  with $d^i : \hom^i(X,Y) \to \hom^{i+1}(X,Y)$

  and $d^2 \; (= d^{i+1}\circ d^i) = 0$

  (3) $\forall X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$ a composition map (a map of cochain cpxs)

$$\hom^{\cdot}(Y,Z)\otimes_k \hom^{\cdot}(X,Y) \longrightarrow \hom^{\cdot}(X,Z)$$

  Naturally a cochain complex

  ℓ^{th} graded component: $\bigoplus_{i+j=\ell} \hom^i(Y,Z)\otimes_k \hom^j(X,Y)$

  differential $d(a^i\otimes b^j) = da^i\otimes b^j + (-1)^i a^i\otimes db^j$

Satisfying

  (1) $\exists$ a unit in $\hom^0(X,X)$ $\forall X$

  i.e. if e_X is the unit, and μ the composition map

$$\mu(e_X\otimes f) = f \quad \forall f\in \hom^{\cdot}(Y,X)$$

$$\mu(g\otimes e_X) = g \quad \forall g\in \hom^{\cdot}(X,Y)$$

  (2) Composition is associative

  i.e. commutativity of:

$$\begin{tikzcd}[column sep=tiny] & \hom(Z,W)\otimes_k \hom(Y,Z)\otimes_k \hom(X,Y) \arrow[dl, "\mu\otimes 1"'] \arrow[dr, "1\otimes\mu"] & \\ \hom(Y,W)\otimes_k \hom(X,Y) \arrow[dr, "\mu"'] & & \hom(Z,W)\otimes \hom(X,Z) \arrow[dl, "\mu"] \\ & \hom(X,W) & \end{tikzcd}$$

- **Remark**: Given a dg category $\mathcal{C}$, one can create its homotopy category notated

$$H^0(\mathcal{C}) \quad (\text{or } H_0(\mathcal{C}), \text{ or } ho(\mathcal{C}), \text{ or } ho\,\mathcal{C})$$

with

$$\mathrm{ob}\,H^0(\mathcal{C}) = \mathrm{ob}\,\mathcal{C}$$

(enriched in k-mod)

$$\hom_{H^0(\mathcal{C})}(X,Y) := H^0\hom^{\cdot}_{\mathcal{C}}(X,Y)$$

with induced composition

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- Example: $\mathcal{C}$ = category of k-cochain complexes

  i.e. $\operatorname{ob}\mathcal{C} = \{k\text{-cochain cpxs}\}$

  $$\hom^i(X,Y) = \{\text{degree } i \text{ linear maps } X \to Y\}$$
  $$= \{\forall j \text{ a linear map } X^j \to Y^{j+i}\}$$

  * Note: not necessarily cochain maps

  with
  $$d^i : \hom^i(X,Y) \longrightarrow \hom^{i+1}(X,Y)$$
  given by
  $$(d^i f)(x) = d_Y f(x) - (-1)^{|f|} f(d_X x)$$

  (Claim: Check this forms dg category with obvious composition)

  Rmk: $H^0 \hom(X,Y) \cong \{\text{cochain maps } X \to Y\} / \text{homotopy}$

- Question: So can we define some dg category where objects are $L_i \subset M$ and $\hom^*(L_i, L_j) = CF^*(L_i, L_j)$

  Answer:

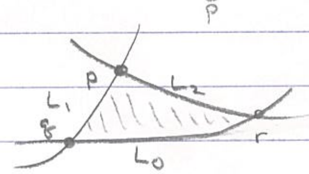
  NO! (Although we can if we loosen things so $\hom^\cdot(L_i, L_j) \simeq CF^*(L_i, L_j)$)

  $\uparrow$ quasi-isomorphism

- Question: What algebraic structure actually pops out?

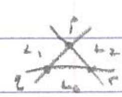
  Answer: A_∞ category

- Exploration: To make a category, we need a map
  $$\underset{\substack{\psi\\ p}}{CF^\cdot(L_1, L_2)} \otimes \underset{\substack{\psi\\ q}}{CF^\cdot(L_0, L_1)} \longrightarrow \underset{\substack{\psi\\ r}}{CF^\cdot(L_0, L_2)}$$

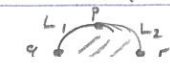


  $$p \otimes q \longmapsto \sum_{r \in L_0 \cap L_2} a_r r$$
  $a_r = \#$ holo^c triangles

  (define on generators)

Defn: Given p, q, r as above, define
$$n_{pqr} = \#\{\text{holo}^c \text{ triangles w/ } \partial \text{ conditions}$$  $\}$

More rigorous:
$$\#\{u : \mathbb{D}^2 \setminus \{1, i, -1\} \to M \text{ satisfying } \partial \text{ conditions and } J\text{-holo}^c\}$$



with some decay conditions, etc...

- Note: From count as defined above, get

$$\mu^2 : CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$$
$$p \otimes q \longmapsto \sum n_{pq}^r \cdot r$$

Two things to check:

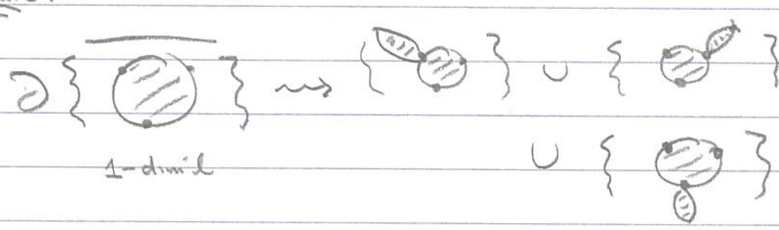
(1) Map of chain complexes

(2) Associative

(Let $\mu^1 : CF(L_i, L_j) \circlearrowleft$ be the differential)

Let's check (1): Need $\mu^1(\mu^2(p \otimes q)) = \mu^2(\mu^1(p) \otimes q) \pm \mu^2(p \otimes \mu^1(q))$

Picture:



(exactness $\Longrightarrow$ no disc/sphere bubbles)