EXR

09/11/15

L4

- Let M be a cmpct mf, $\partial M = \phi$. Show that if any of M's H^*_{dR} (for $0 \le * \le \dim M$) vanish, M cannot be symplectic.
- Let $M = T^*Q$, Q – C^∞ mf. Given any $Z \subset Q$ smooth submf, define $T^*_Z Q := \{(z,\alpha) : z \in Z, \alpha \in T^*Q|_Z, \alpha|_{TZ} \equiv 0\}$. Show that this is an exact Lagr. submf.
- Fix (M,ω). Prove that a compatible J exists. Prove that the space of such J is contractible.

Last time: motivated the choice of (M,ω) s.t. $\omega = d\theta$, $L_i \subset M$ s.t. $\theta|_{L_i} = df_i$, $f_i : L_i \to \mathbb{R}$.
(these assumptions make the square of Morse diff'l equal to 0)

Thrm $\forall$ such L_i, under good conditions – can define a cochain cx $CF^*(L_i, L_j)$ – Floer cochain cx

Can think of it as of a dg cat: $\mathcal{O}b = L_i$, $\mathcal{M}or = CF^*$.

Recall the def. of a dg cat:

Def For a ring k, a dg category $\mathcal{C}$ over k is such a cat:
- a class of objects $\mathcal{O}b\, \mathcal{C}$
- $\forall$ pair of objects $X, Y \in \mathcal{O}b\, \mathcal{C}$ – a cochain cx $\mathrm{Hom}^\bullet(X,Y)$
- $\forall X, Y, Z \in \mathcal{O}b\, \mathcal{C}$ – a map of k-modules, cochain cxs
$\mathrm{Hom}^\bullet(Y,Z) \otimes \mathrm{Hom}^\bullet(X,Y) \longrightarrow \mathrm{Hom}^\bullet(X,Z)$ $\quad d(f\circ g) = df \circ g + (-1)^{\deg f} f \circ dg$
called the composition

Satisfying:
- $\forall X \in \mathcal{O}b\, \mathcal{C}$ $\exists$ a unit $id_X \in \mathrm{Hom}^0(X,X)$ — it is also denoted by e_X
- composition is associative

__Comment__: e_X is unit means: $\forall f: X \to Y: \ f \circ e_X = f$
$\forall g: Z \to X: \ e_X \circ g = g$

associativity means:

$$W \leftarrow Z \leftarrow Y \leftarrow X$$

$$\begin{tikzcd}
& \operatorname{Hom}^\bullet(Z,W) \otimes \operatorname{Hom}^\bullet(Y,Z) \otimes \operatorname{Hom}^\bullet(X,Y) \arrow[dl, "\circ \otimes id"'] \arrow[dr, "id \otimes \circ"] & \\
\operatorname{Hom}^\bullet(Y,W) \otimes \operatorname{Hom}^\bullet(X,Y) \arrow[dr, "\circ"'] & \circlearrowleft & \operatorname{Hom}^\bullet(Z,W) \otimes \operatorname{Hom}^\bullet(X,Z) \arrow[dl, "\circ"] \\
& \operatorname{Hom}^\bullet(X,W) &
\end{tikzcd}$$

__Rmk__ Given a dg cat $\mathcal{C}$, one can create its "__homotopy cat__", i.e.:

$H^0(\mathcal{C})$, $Ho(\mathcal{C})$, $ho(\mathcal{C})$ ← different notation for one category

$\operatorname{Ob} H^0(\mathcal{C}) = \operatorname{Ob} \mathcal{C}$

$\operatorname{Hom}_{H^0(\mathcal{C})}(X,Y) = H^0(\operatorname{Hom}^\bullet_{\mathcal{C}}(X,Y))$

Composition induced.

__Good exr.__: Check that $H^0(\mathcal{C})$ is well-defined.

__Exr.__ $\mathcal{C} :=$ cat. of k-cochain cxs

$\operatorname{Hom}^i(X,Y) = \{\deg i \ k\text{-linear maps } X \to Y\} = \left\{ \begin{tikzcd}[row sep=small]
& \vdots \\
\vdots & Y^{l+i+1} \arrow[u] \\
X^{l+1} \arrow[u] \arrow[ur, "+i"] & Y^{l+i} \arrow[u] \\
X^{l} \arrow[u] \arrow[ur, "+i"] & \vdots \arrow[u] \\
\vdots \arrow[u] & Y^{l} \arrow[u] \\
{} \arrow[ur, "+i"] &
\end{tikzcd} \right\}$

$\ \ \ \ \psi$

$f \rightsquigarrow df = d_Y \circ f + (-1)^{\deg f} f \circ d_X$

Attempt to define Fukaya category:

So, can we define some dg cat, where objects are $L_i \subset M$,

$$\mathrm{Hom}^\bullet(L_i, L_j) = CF^*(L_i, L_j)\ ?$$

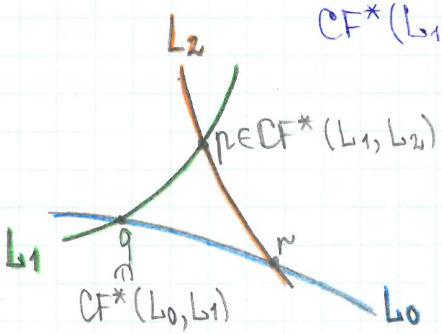
ANSWER: NO.

We can if we loosen things s.t. $\mathrm{Hom}^\bullet(L_i, L_j) \underset{\text{quiso}}{\simeq} CF^*(L_i, L_j)$.

What alg str're really appears is that of A_∞ cat.

EXPLORATION: to make a cat, we need ~~to~~ a map

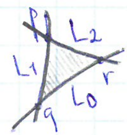
$$CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$$



$p \otimes q \overset{?}{\longmapsto} \sum a_r r$

idea: "count Δ's"

Def. Given p, q, r, define $n_{pq}{}^r := \#\{$holo Δ's in M with bdry conditions:



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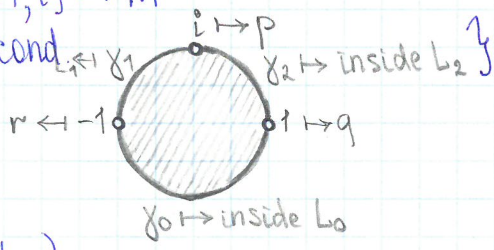
More rigorously:

$n_{pq}{}^r := \#\{$holo $u : D^2 \setminus \{\pm 1, i\} \to M$ satisfying bdry cond $\}$



So define "composition" by:

$$\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2) \quad \text{by } p \otimes q \mapsto \sum_{r \in L_2 \cap L_0} n_{pq}{}^r \cdot r$$

Two things to check: (1) map of chain cxs

(2) associativity $\rightsquigarrow$ FAIL

Let μ^1 define the differential:

$$\mu^1 : CF^*(L_i, L_j) \longrightarrow CF^{*+1}(L_i, L_j)$$

Need to show in (1): $\mu^1(\mu^2(p \otimes q)) =$

$$= \mu^2(\mu^1(p) \otimes q) \pm \mu^2(p \otimes \mu^1(q))$$

Can be shown by drawing pictures.