Lecture

Last Time

Setup: $(M, \omega = d\theta)$ $\quad L_i \subset M$ exact $\quad i = 0, \dots, k$

We examined:

- $\mathcal{R} = \mathcal{R}_{k+1} = \{\text{space of holom. str. on } \mathbb{D}^2 \setminus \{k+1 \text{ pt. on } \partial\}\}$

  $= \mathrm{Conf}_{k+1}(\partial\mathbb{D}^2) / PSL(2,\mathbb{R})$

  $\dim \mathcal{R}_{k+1} = k-2$

- $\{(u, S) \mid S \in \mathcal{R}_{k+1},\ u : S \to M$ satisfying
  - $\gamma_i \subset \partial S,\ u(\gamma_i) \subset L_i$
  - $(du - X_Y)^{0,1} = 0$

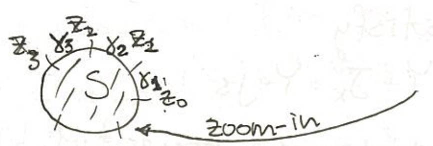
  $\uparrow$ perturbing $(du)^{0,1} = 0$

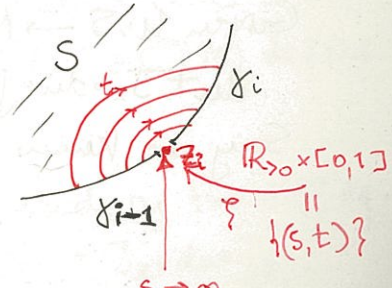
  $Y \in \Omega^1_{dR}(S; C^\infty(M))$

  $X_Y \in \Omega^1_{dR}(S; u^*TM)$

  $TS \ni v \longmapsto X_{Y(v)}(u(x))$

<u>Rmk</u> This also resolves issue of $L_i \not\pitchfork L_{i+1}$.





- Choosing holom. parametrization:

  $\zeta : \underset{s}{\mathbb{R}_{>0}} \times \underset{t}{[0,1]} \longrightarrow S$ near z_i.

  we demand that as $s \to \infty$

  ζ converges to a Hamiltonian chord

  $\uparrow$ perturbs $u(z_i) \in L_{i-1} \cap L_i$

By a Hamiltonian chord I mean a C^∞ map:

$C : [0,1] \longrightarrow M$ s.t. $C(0) \in L_{i-1}$ $C(1) \in L_i$ and

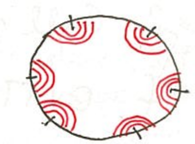
$\dot{C}(t) = X_Y(C(t))$

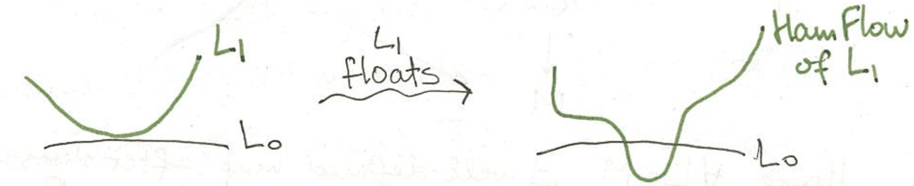
⚠ : When choosing Y, choose Y to be independent of s, $s \in \mathbb{R}_{>0}$.

<u>Thm</u> (Last time) $\exists$ plenty of choices of (J_x, Y) making this moduli space of $\{(u, S)\}$ a C^∞-mfld.
(J_x is a choice of almost $\mathbb{C}$ str. on M $\forall x \in S$)

<u>Rmk</u> To choose ζ concretely:

$$\begin{array}{ccccc}
S & \longrightarrow & \mathcal{S} & \longleftarrow & \zeta : \mathbb{R} \times [0,1] \times \mathcal{R}_{k+1} \\
\downarrow & & \downarrow & \swarrow_{\text{project}} & \\
\{S\} & \xhookrightarrow{S} & \mathcal{R}_{k+1} & & \\
\| & & & & \\
* & & & &
\end{array}$$





<u>Today</u>: What is the <u>dimension</u> of $\{(u, S)\}$?

<u>Idea</u> Given a map $\mathbb{D}^2 \xrightarrow{u} M$, s.t. $\partial\mathbb{D}^2 \xrightarrow{u} \mathrm{LagGr}(M$

i.e. $\forall z \in \partial\mathbb{D}^2$ is assigned a (specified) Lagrangian subspace of $T_{u(z)}M$

$U(n)/O(n)$ for $\dim M = 2n$

$$\begin{tikzcd} \mathrm{LagGr}(T_xM) \arrow[r] \arrow[d] & \mathrm{LagGr}(M) \arrow[d] \\ \{x\} \arrow[r, "x"] & M \end{tikzcd}$$

$\mathrm{LagGr}(T_xM) \cong U(n)/O(n)$; ← fiber bundle over M

Ex If you have map $(\mathbb{D}^2, \partial\mathbb{D}^2) \xrightarrow{u} (M, L)$ then u defines a map $\partial\mathbb{D}^2 \to \mathrm{LagGr}(M)$. Then the "winding number" of $\partial\mathbb{D}^2 \to \mathrm{LagGr}(TM)$ determines the dimension of $\{(u,S)\}$. The winding # is called the **Maslov** index of u.

"winding number": $\pi_1(U(n)/O(n)) \cong \mathbb{Z}$ realized by

$$U(n)/O(n) \xrightarrow{\det^2} S^1$$

Defn Call a symplectic mfld M <u>almost Calabi-Yau</u> if $c_1(TM) = 0 \in H^2(M, \mathbb{Z})$.

Consequence:

$$\begin{tikzcd} \mathrm{LagGr}(M) \arrow[r, "\det^2"] \arrow[d] & S^1 \times M \arrow[dl] \\ M & \end{tikzcd}$$

Hence $\forall L \subset M$ $\exists$ well-defined map (after choosing trivial.) $L \longrightarrow S^1$. Suppose $\exists$ a lift $\alpha : L \dashrightarrow \mathbb{R}$,

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ L \arrow[ur, dashed, "\alpha"] \arrow[r] & S^1 \end{tikzcd}$$

Defn α is called a <u>grading</u> on L.

Rmk The $\mathbb{Z}$ worth of choices of α corresponds to shifts of L as an object of $\mathrm{Fuk}(M)$.

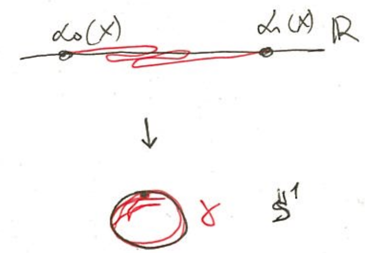
The grading on $CF^\circ(L_0, L_1)$

Suppose (L_0, α_0) & (L_1, α_1) are graded Lagrangians. Fix $x \in L_0 \cap L_1$. The degree of x can be defined by:

At $x \in M$ we have:

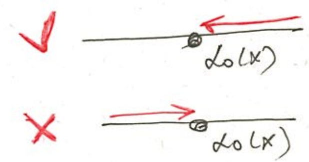
$$\begin{tikzcd} \widetilde{\mathrm{LagGr}}(T_xM) \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ \mathrm{LagGr}(T_xM) \arrow[r, "\det^2"] & S^1 \end{tikzcd}$$

and $\alpha_0(x), \alpha_1(x) \in \mathbb{R}$ give elements $\widetilde{T_xL_0}, \widetilde{T_xL_1} \in \widetilde{\mathrm{LagGr}}(T_xM)$



Want to say: grading is just a winding number of γ, but!

- α_0 and α_1 may go to different points; Here's how we deal with it: Look at a path $\tilde{\gamma}$, that begins at $\alpha_1(x)$ and ends at $\alpha_0(x)$ with negative derivative:



Count the signed intersection of $\tilde{\gamma}$ with either
- $\alpha_1(x) + \mathbb{Z}$

or
- $\alpha_0(x) + \mathbb{Z}$ ← «PROBABLY THIS ONE»