(Math 277)

9/21

Survey?
D. Faust, Report on Sexual Assault (7 PM SC Hall D).

Last time:

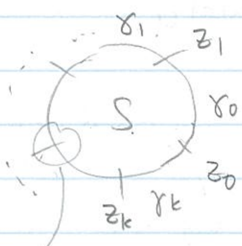
Setup: $(M, \omega = d\theta)$ symplectic mfld.
$L_i \subset M$, $i = 0, \dots, k$ exact.

We examined

- $R = R_{k+1} = \{$space of holom. str. on $D^2 \setminus \{k+1\ \partial$ pts$\}\}$
$= \mathrm{Conf}_{k+1}(\partial D^2) / PSL(2, \mathbb{R})$.

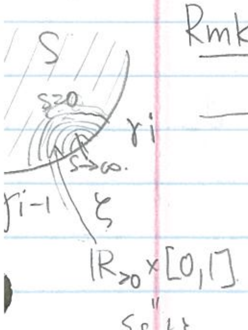
dim $R_{k+1} = k-2$.

- $\{(u, S) \mid S \in R_{k+1},\ u : S \to M$ satisfying:



- $\gamma_i \subset \partial S$.
$u(\gamma_i) \subset L_i$
- $(du - X_Y)^{0,1} = 0$

perturbation of $(du^{0,1}) = 0$.



Rmk This also resolves the issue of $L_i \not\pitchfork L_{i+1}$.

- Choosing holomorphic parametrizations $\xi : \mathbb{R}_{>0} \times [0,1] \to S$, we demand that as $s \to \infty$, ξ converges to a Hamiltonian chord.

perturbs $u(z_i)$ in $L_{i-1} \cap L_i$.

X_Y?

- $Y \in \Omega^1_{dR}(S; C^\infty(M))$.
- $X_Y \in \Omega^1_{dR}(S; u^*TM)$

$T_x S \ni v_x \longmapsto X_{Y(v_x)}(u(x))$.

By a Hamiltonian chord, Hiro means a C^∞ map

$c : [0,1] \longrightarrow M.$

$$\text{s.t.} \begin{cases} c(0) \in L_{i-1} \\ c(1) \in L_i \\ \dot{c}(t) = X_\varphi(c(t)). \end{cases}$$

⚠ When choosing φ, choose φ to be independent of s, $s \in \mathbb{R}_{>0}$.

<u>Thm</u> (from last time) $\exists$ plenty of choices of (J_x, φ) making the moduli space $\{(u,s)\}$ a <u>C^∞ mfld.</u>

(J_x is a choice of compatible almost cplx structure on M for all $x \in S$.)

– Question about almost $\mathbb{C}$-structure for each S?

<u>Rmk</u> To choose ξ concretely,

$$\begin{tikzcd}
S \arrow[r] \arrow[d] & \mathcal{S} \arrow[d] & \mathbb{R} \times [0,1] \times R_{k+1} \arrow[l, "\xi"'] \arrow[dl, "\text{project.}"] \\
* = \{S\} \arrow[r, hook, "S"] & R_{k+1} &
\end{tikzcd}$$

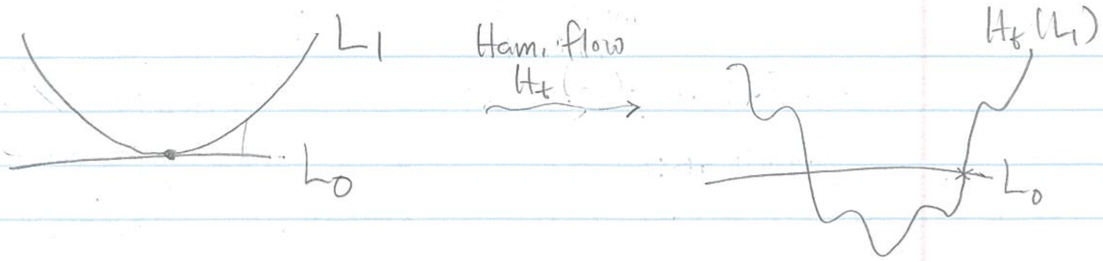
These ξ are called <u>strip-like ends</u>.

–

Rmk. Given $\xi : \mathbb{R} \times [0,1] \longrightarrow S$.

$$\dot{c}(t) = \xi^* X_\varphi \left(\frac{\partial}{\partial t}\right) \in T(\mathbb{R}\times[0,1]).$$

Q: Alt. motivation for perturbation. (Y?)



Want to replace nontransverse intersection w/ pt after Ham. flow $\leftarrow$ this is a Hamiltonian chord.

For patching, need to choose global $Y \in \Omega^1(S; C^\infty(M))$.

Today: What is the dimension of $\{(u, S)\}$?

Idea: given a map $D^2 \xrightarrow{u} M^{2n}$ s.t.

$$\partial D^2 \xrightarrow{u} \text{LagGr}(M).$$

i.e. $\forall z \in \partial D^2$, ~~the tangent vector to ∂D^2 at~~ z is assigned a ~~(specified)~~ Lagrangian subspace of $T_{u(z)}M$

$U(n)/O(n)$
$\cong$ fiber bundle$/M$

$$\begin{tikzcd} \text{LagGr}(T_xM) \arrow[r] \arrow[d] & \text{LagGr}(M) \arrow[d] \\ \{x\} \arrow[r, hook, "x"] & M \end{tikzcd}$$

Ex. If you have a map $(D^2, \partial D^2) \xrightarrow{u} (M, L)$, then u defines a map $\partial D^2 \longrightarrow \text{LagGr}(M)$.

Then the "winding number" of

$$\partial D^2 \longrightarrow \text{LagGr}(M)$$

determines the dimension of $\{(u, S)\}$.

why?
$\boxed{\text{spectral flow}}$
(how e-vals change)
&
index of some operator.
↑
(Cauchy-Riemann)

This winding # is called the Maslov index of u.

Concretely, recall $\pi_1(U(n)/O(n)) \cong \mathbb{Z}$, realized by $U(n)/O(n) \xrightarrow{\det^2} S^1$.

(To see this, look in H^* and use pairing).

Defn. Call a symplectic mfld <u>almost Calabi-Yau</u> if $c_1(TM) = 0$.

(Recall prev. exercise on compatible $\mathbb{C}$-structures on TM - contractible choice)

<u>Consequence:</u> $\exists$ choice of trivialization $\det(TM) \cong \mathbb{C}\times M$.

$$\begin{tikzcd}
 & L \arrow[r, mapsto] & \in S^1 \\
U(n)/O(n) \arrow[r] \arrow[d] & \mathrm{LagGr}(M) \arrow[r, "\det^2"] \arrow[d] & S^1 \times M \arrow[dl] \\
pt \arrow[r] & M &
\end{tikzcd}$$

$\Rightarrow$ $\forall L \subset M$ $\exists$ well-defined map (after choosing trivialization) $L \to S^1$

Suppose $\exists$ a lift.

$$\begin{tikzcd}
 & \mathbb{R} \arrow[d] \\
L \arrow[r] \arrow[ur, dashed, "\alpha"] & S^1
\end{tikzcd}$$

Defn: α is called a <u>grading</u> on L

<u>Rmk</u> The $\mathbb{Z}$-worth of choices of α corresponds to <u>shifts</u> of an object of $\mathcal{F}uk(M)$.

like in a Δ'ed cat.

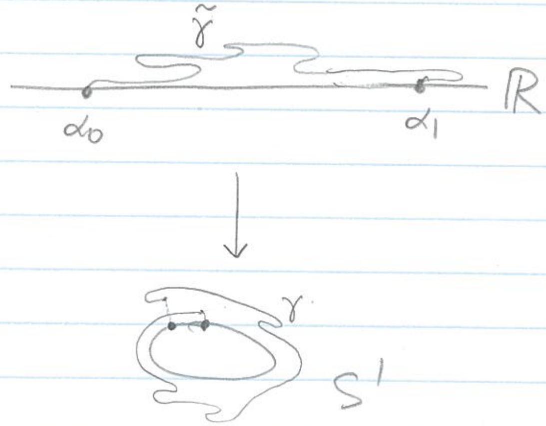
The grading on $CF^*(L_0, L_1)$

Suppose (L_0, α_0) and (L_1, α_1) are graded Lagrangians. Fix $x \in L_0 \cap L_1$. The degree of x can be defined by

At $x \in M$, we have

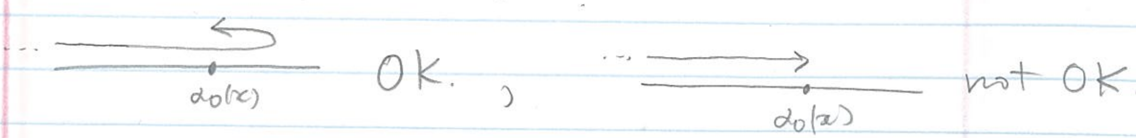
$$\begin{tikzcd}
\widetilde{LagGr(TM)} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\
LagGr(TM) \arrow[r, "\det^2"'] & S^1
\end{tikzcd}$$

and $\underbrace{\alpha_0(x), \alpha_1(x)}$ $\in \mathbb{R}$ give el'ts $\widetilde{T_x L_0}, \widetilde{T_x L_1}$ in $\widetilde{LagGr(TM)}$.



But α_0, α_1 not nec. in same fiber!

Convention: Look at a path $\tilde{\gamma}$ that begins at $\alpha_1(x)$ and ends at $\alpha_0(x)$ w/ <u>negative</u> derivative
(up to $\pm$)



(maybe need similar cond^n at $\alpha_1(x)$.)

Count the signed intersection # of $\tilde{\gamma}$ with either

- $\alpha_1 + \mathbb{Z}$, or
- $\alpha_0 + \mathbb{Z}$ (one of these is correct).

↖ probably this one.

<u>Next time</u>: • how to compute dim
• why $\mu^k: \cdots \otimes \cdots \longrightarrow \hom(\cdots)[2-k]$