Lecture

Dimension of $\mathcal{M} = \{(u,S)\}$

Last time Given two graded Lagrangians

$L_i = (L_i, \alpha_i)$ $i = 0, 1$

$$\alpha_i : \begin{tikzcd} L_i \arrow[r] \arrow[dr, "\det^2"'] & \mathbb{R} \arrow[d, "\pi"] \\ & S^1 \end{tikzcd}$$

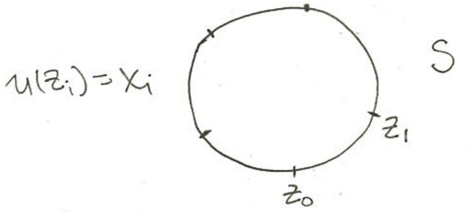
Goal of today: Sketch an arguement for the following

claim: Fix $(L_i, \alpha_i) \subset M$ $i = 0, 1, \dots, k$. And $x_i \in L_{i-1} \cap L_i$

Let $(u,S) \in \mathcal{M} \leftarrow$ maps: $u : S \to M$ w/ $\partial u \subset \bigcup_{i=0}^{K} L_i$,

marked pts $\to x_i$, $(u - X_\gamma)^{0,1} = 0$

Then $\dim \mathcal{M} = \underset{\text{incoming}}{\underset{\uparrow}{i(x_0)}} - \underset{\text{outcoming}}{\underset{\uparrow}{\sum}} i(x_i) + (K-2)$



Label z_0 as "incoming" ∂ pt.

$z_1 \dots z_K$ as "outcoming" ∂ pt.

Let $i(x_i) :=$ [the integer from the last class]

$\uparrow$

signed intersection of $\tilde{\gamma}$ with $\alpha_i(x) + \mathbb{Z}$. or "almost winding number"

$x_0 \in CF(L_0, L_k)$

$x_1 \in CF(L_0, L_1)$

$\mu^k : x_k \otimes \dots \otimes x_1 \longmapsto \sum \# \cdot x_0$

i.e. $\mathcal{M}_u \subset \mathcal{M}$ - component containing u

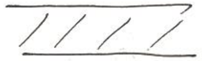
$\dim \mathcal{M}_u = 0 \iff i(x_0) = \sum_{\text{outgoing}} i(x_i) + 2 - K$

this is why μ^k is a map.

$$CF(L_{k-1}, L_k) \otimes \dots \otimes CF(L_0, L_1) \longrightarrow CF(L_0, L_k)[2-K]$$

Def: As a graded ab. gp : $CF^*(L_0, L_1) = \bigoplus_{x_i} \mathbb{Z}[-i(x_i)]$.

Let's first examine $S = \mathbb{R} \times [0,1]$



Fix a map $u : S \to M$ satisfying ∂ conditions, pseudoholomorphic $((du - X_\gamma)^{0,1} = 0)$

Last time: I claimed you could compare $\dim \mathcal{M}_u$ by examining some winding number $u|_{\partial S} : \partial S \to \mathrm{LagGr}$

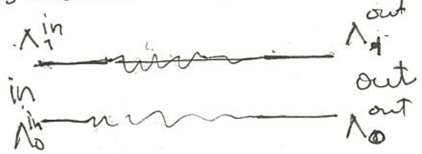
$$\begin{tikzcd} \mathbb{R} \times \{1\} \arrow[r] & \mathrm{LagGr}(u^*TM) \arrow[r, phantom, "\cong"] & S \times \mathrm{LagGr}(\mathbb{C}^n) \arrow[dr] & \\ & & & \mathrm{LagGr}(\mathbb{C}^n) \\ \mathbb{R} \times \{0\} \arrow[r] & \mathrm{LagGr}(u^*TM) \arrow[r, phantom, "\cong"] & S \times \mathrm{LagGr}(\mathbb{C}^n) \arrow[ur] & \end{tikzcd}$$

How to make a loop in LagGr?

a Assume the maps $\mathbb{R} \times \{i\} \longrightarrow \mathrm{LagGr}(\mathbb{C}^n)$ compactly supported. This determines:

$\Lambda_0^{in}, \Lambda_1^{in}$

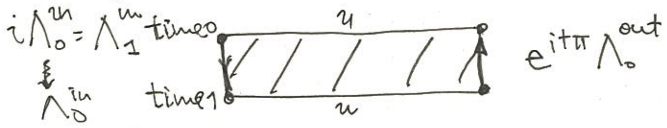
$\Lambda_0^{out}, \Lambda_1^{out} \in \mathrm{LagGr}(\mathbb{C}^n)$



Can find a trivialization of the bundle $\mathbb{C}^n \times S$

$$\Lambda_1^{in} = i \Lambda_0^{in} \quad \text{and} \quad \Lambda_1^{out} = i \Lambda_0^{out}$$

This defines a loop in $\mathrm{LagGr}(\mathbb{C}^n)$:

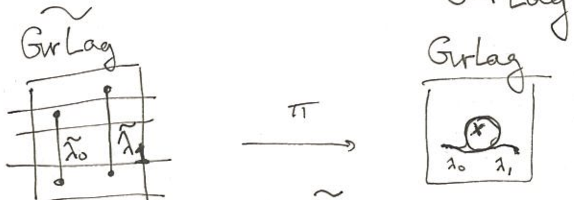


Thm (Floer) $\dim \mathcal{M}_u$ = winding # of this path

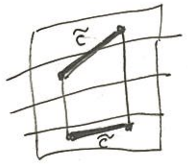
How to relate to $i(x_{in})$, $i(x_{out})$?

Assume we choose lift:

$$\begin{tikzcd}
\tilde{\lambda}_i : \mathbb{R}\times\{i\} \arrow[r] \arrow[dr, "u"'] & \widetilde{GrLag} \arrow[d, "\pi"] \\
 & GrLag
\end{tikzcd}$$



We can choose $\underbrace{\text{path in}}_{}^{\tilde{c}}$ $\widetilde{GrLag}$ connecting the endpoints of $\tilde{\lambda}$.



loop in $\widetilde{GrLag}$

$\Downarrow$

contractible loop in GrLag

So computing the bdry #'s given by λ's is the same as computing winding #'s given by the $\tilde{c}$.

But the indices $i(x_i)$ were defined to compute the indicies of c.

(The area of the strip //)

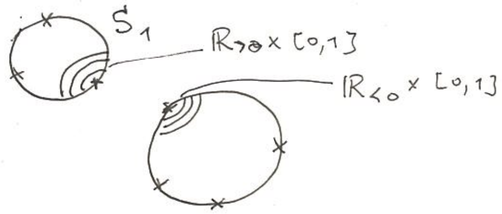
For <u>other cases</u>: "think about" exact same.

"Actual Proof" gluing $\bar\partial$ operators over disks

If $S = S_1 \# S_2$

$\uparrow$ holomorphic connect sum

Then $\text{index}(\bar\partial_S) = \text{index}(\bar\partial_{S_1}) + \text{index}(\bar\partial_{S_2})$



<u>Briefly</u>: When you take the derivative D of $(--X_\gamma)^{0,1}$ get a Fredholm operator (i.e. $\dim\ker D$, $\dim\operatorname{coker} D < \infty$)

Turns out ~~deform~~ we can deform D to standard $\bar\partial$ through Fredholm operators.

<u>Fact</u>: index doesn't change under Fredholm path.

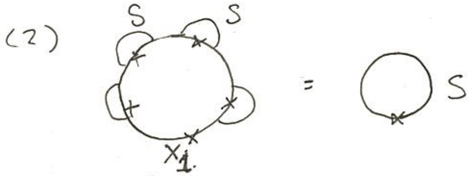
When D is surjective (condition) index = dimker − dimcoker = = dim ker = dim $\mathcal{M}_u$

it follows coker=0 $\overset{?}{\Rightarrow}$

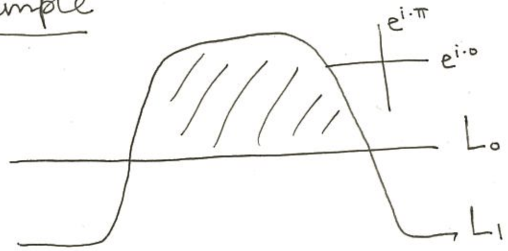
Gives the gluing formula:

(1) Can compute index for $S = \mathbb{D}^2 \setminus$ single pt.

(2)



<u>Example</u>



WILL COME
BACK TO IT
NEXT TIME

$M = \mathbb{C}$, $dx \wedge dy$, $J = i$

Choose trivilization of $\det^2(TM)$ so that ——— $1 \in S^1$ | $i \in S^1$