Today: Examples.

Background: Given (M, ω) $\omega = d\theta$. (symplectic)
can we find $M^\vee$ s.t.

$$\mathrm{Fuk}(M) \cong \underset{\substack{\uparrow \\ \text{dg enhancement}}}{D^b} \mathrm{Coh}(M^\vee)$$

Ex (0) $M = pt$.
ob $\mathrm{Fuk}(M) = ?$

$TM \cong pt$
$\det(TM) \cong \mathbb{R} \times M$

The 2-form $\omega = 0$ is a symplectic form.

Lagrangians: $L = pt$. $\left(\begin{matrix} \dim L = \frac{1}{2} \dim M \\ \omega|_L = 0 \end{matrix} \right)$

$L = \emptyset$ $\dim \emptyset$ is ill-defined
It's a manifold of every dimension

By grading, what do we mean?

Let's just mean, by grading, we choose an integer when $L = pt$.
When $L = \emptyset$, $\exists!$ map from $\emptyset$ to anything, so $\exists!$ grading on $\emptyset$.

Based on what we have discussed so far, an object of $\mathrm{Fuk}(M)$ is a pair

$$(\underset{\text{Lag}}{L}, \underset{\text{grading}}{\alpha})$$

$$\mathrm{ob}\,\mathrm{Fuk}(pt) \cong \mathbb{Z} \sqcup \{\emptyset\}.$$

$$(pt, n) \longleftarrow n$$

$$(\emptyset, \text{unique grading}) \longleftarrow \{\emptyset\}$$

Remark on grading:

As for grading, recall that a grading for L is a lifting

$$\begin{tikzcd}
 & & \mathbb{R} \arrow[d] \\
 & GrLag(n) \arrow[r] \arrow[d] & S^1 \\
L \arrow[ur] \arrow[r, hook] \arrow[uurr, bend left, "\text{grading of } L"] & M \cong pt &
\end{tikzcd}$$

Compute $\hom_{\mathrm{Fuk}(pt)}(L_0, L_1)$

If L_0 or $L_1 = \emptyset$,

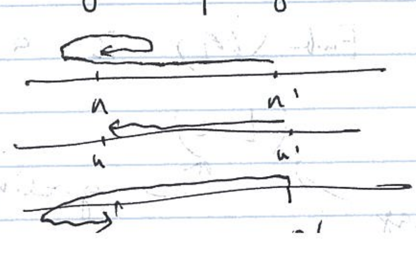
$$CF^*(L_0, L_1) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z} \cong 0$$

Now we only need to compute

$$CF^*(pt(n), pt(n'))$$

$$\cong \mathbb{Z}[-n'+n].$$

Recall the degree of generators of CF^* is defined by



or

not

$$\mathbb{R} \downarrow S^1 \ni 1 = e^{0i}$$

but this time, $n-n' \in \mathbb{Z}$ because it's zero dimensional,
so the degree is just the winding number $(n-n')$.

Now compute μ^1, μ^2 etc.

$\mu^1 = 0$ because our graded ab. gp. has no other choice!

Or, the formal dimension of the moduli space of holo. strips is (-1).

$$\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \longrightarrow CF^*(L_0, L_2)$$

i.e.

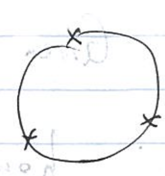
$$\mathbb{Z}[-n_2+n] \otimes \mathbb{Z}[-n_1+n_2] \longrightarrow \mathbb{Z}[-n_1+n]$$

Claim: This is the identity map. (a.k.a. the multiplication map)

Pf: Given generators

$$x_{12} \in CF(L_1, L_2)$$
$$x_{01} \in CF(L_0, L_1)$$
$$x_{02} \in CF(L_0, L_2)$$

Count $\# \{(u, s)\} \in \mathcal{M}$

 $\xrightarrow{u} M.$

The map u must be constant, S is unique.

$\# \{(u, s)\} = 1.$

So $\mu_2(x_{12}, x_{01}) = x_{02}$.

Claim: $\mu^k = 0 \quad \forall k \geq 3$.

In fact, $\mathcal{M} = \{(u, s)\}$

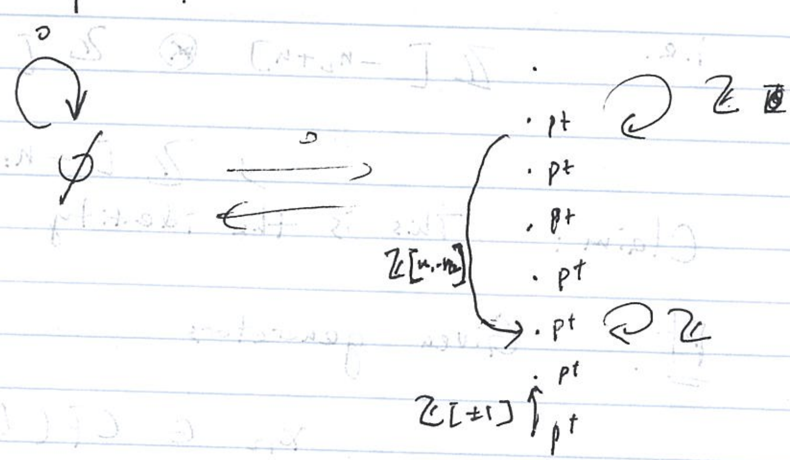
$\exists!$ — space of holo. structures on the disk with $(k+1)$ punctured points on boundary.

So $\mathcal{M}$ has no zero dim component.

We can also see this for degree reasons.

What is the category?

objects and morphisms:



This couldn't possibly be $D^b Coh(X)$ for any X.

Why not? (1) $D^b Coh(X)$ has $\oplus$

Given any E, F $\exists$ object $E \oplus F$

s.t. $hom(E \oplus F, -)$
$\cong$
$hom(E, -) \times hom(F, -)$

however, there is no 2-dim hom space in $Fuk(pt)$.

For this reason, we consider a "completion" of $Fuk(M)$
(When $M = pt$, and when M = anything)

Philosophy: Any property that a math object Y has, the object $Map(X, Y)$ inherits it for $\forall X$. (example: if Y is an Abelian group, so is $Map(X, Y)$)

The properties that $D^k Ch(\sim)$ has (algebraic properties) are the properties that $Chain_{\mathbb{Z}}$ has.

If we want to construct somethings with the same algebraic properties, using $Fuk(M)$, consider

$$F_{A_\infty}(Fuk(M)^{op}, Chain_{\mathbb{Z}})$$

Lemma (Yoneda) $\exists$ a fully faithful embedding $Fuk(M) \hookrightarrow Fun_{A_\infty}(Fuk(M)^{op}, Chain_{\mathbb{Z}})$

<u>Remark</u> Fully faithful means that means that the image of functor is a copy of $Fuk(M)$ itself.

~~Rmk~~ This holds ~~holds~~ for any A_∞ cat. $\mathcal{C}$.
not just Fukaya categories.

The functor is

$$\mathcal{C} \longrightarrow \mathrm{Fun}_{A_\infty}(\mathcal{C}^{op}, \mathrm{Chain}_{\mathbb{Z}})$$

$$L \longmapsto \hom_{\mathcal{C}}(-, L).$$

Rmk. A functor of A_∞ cat. should consists of

- a functor $\mathrm{ob}\,\mathcal{C} \xrightarrow{F} \mathrm{ob}\,\mathcal{D}$
- a linear map $\hom_{\mathcal{C}}(L_0, L_1) \to \hom_{\mathcal{D}}(F(L_0), F(L_1))$
- a bunch of other maps saying the above data respects composition up to ~~h.~~ homotopies (of homotopies, of homotopies ...)

The functor $\hom_{\mathcal{C}}(-, L) : \mathcal{C}^{op} \to \mathrm{Chain}_{\mathbb{Z}}$.

Given $X \in \mathrm{ob}\,\mathcal{C}$

$$X \mapsto \hom_{\mathcal{C}}(X, L).$$

on morphisms

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[dr, "g\circ f"'] & X_1 \arrow[d, "g"] \\ & \end{tikzcd}$$

Given $f \in \hom_{\mathcal{C}}(X_0, X_1)$

we obtain a map

$$\hom_{\mathcal{C}}(X_1, L) \longrightarrow \hom(X_0, L)$$

$$g \longmapsto g \circ f$$

Remark: If $\mathcal{C}, \mathbb{D}$ are dg-cat.

$$\mathrm{Fun}_{A_\infty}(\mathcal{C}, \mathbb{D}) \not\simeq \mathrm{Fun}_{dg}(\mathcal{C}, \mathbb{D})$$

This is good

Think of this as wrong.