Lecture

Today: examples

Background: Given $(M, \omega = d\theta)$ can we find $M^\vee$ s.t.

$Fuk(M) = D^b Coh(M^\vee)$ ← by dg enhancement

Ex (0) $M = pt$

$TM = pt$ (rank 0 v.bundle)

$\det(0) = \mathbb{R}$ ⟹ the 2form $\omega = 0$ is a sympl. form

$L = pt$ $(\dim = 0 = \frac{1}{2}\dim M)$

$L = \emptyset$ ($\dim \emptyset$ is ill defined, can be any.)

By grading what do we mean?

Let's just mean a choice of integers (when $L = pt$)

When $L = \emptyset$ $\exists ! \ \emptyset \to$ anything so $\exists !$ grading on $\emptyset$.

Based on what we've discussed so far, an object of $Fuk(M)$ is a pair: (L, α)

Lag ↗ ↖ grading

$ob Fuk(pt) \cong \mathbb{Z} \amalg \{\emptyset\}$

$(pt, n) \xleftarrow{\text{biject.}} n$

$(\emptyset, \text{unique grad.}) \longleftarrow \emptyset$

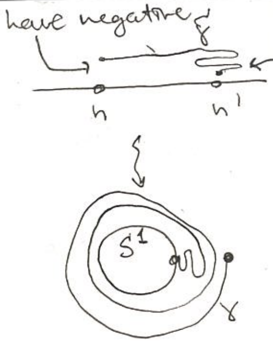
~~If $L_0 \ne L_1$~~ $CF^*(L_0, L_1) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}$

If L_0 or L_1 is $\emptyset$ then $CF^*(L_0, L_1) = 0$

We have to compute: $CF^*(pt(n), pt(n')) \cong \mathbb{Z}[(n'-n)]$

→ path should have negative derivative, when it approaches n

curve has to leave with positive derivative



$\mathbb{R}$

$\downarrow$

$S^1 \ni 1 = e^0$

The winding number is $n' - n$.

Now compute μ^1, μ^2, etc...

$\mu^1 = 0$ because our graded ab gp. has no other choice:

$CF^*(pt[n], pt[n']) =$ $\cdots \to 0 \to 0 \to 0 \to \mathbb{Z} \to 0 \to \cdots$ $\quad n'-n$

Or geometrically: Any holom. strip $u: \mathbb{R} \times [0,1] \to M$ must be constant. Hence $\mathcal{M}_u / \mathbb{R}$ always has dimension -1. So no 0-dimensional boundary components.

$\mu^2 = CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$

$\mu^2: CF^*(pt[n_1], pt[n_2]) \otimes CF^*(pt[n_0], pt[n_1]) \to$

$\to CF^*(pt[n_0], pt[n_2])$

$\mathbb{Z}[-(n_2 - n_1)] \otimes \mathbb{Z}[-(n_1 - n_0)] \longrightarrow \mathbb{Z}[-(n_2 - n_0)]$

Claim: This is identity map

Pf Given generator $x_{12} \in CF(L_1, L_2)$
$x_{01} \in CF(L_0, L_1)$
$x_{02} \in CF(L_0, L_2)$

count $\#\{(u, S)\} \in \mathcal{M}$  $\xrightarrow{u} M$

Since $u = $ const, S is unique $\# = 1$.

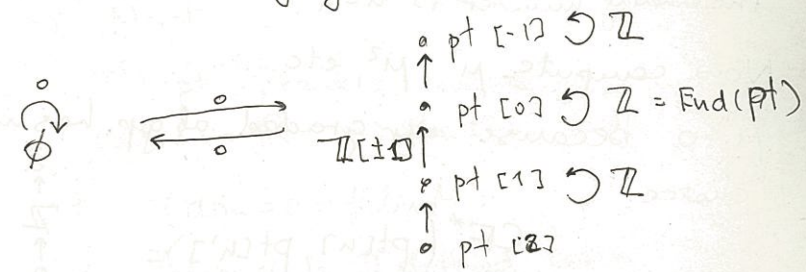
$\mu^2(x_{12}, x_{01}) = x_{02}$

Claim $\mu^k = 0 \;\forall\, k \geq 3$

$\mathcal{M} = \{(u, S)\} \xrightarrow{\exists!} \{S\} \cong \mathbb{R} \implies \mathcal{M} \cong \mathbb{R}$

So $\mathcal{M}$ has no zero dimensional components.

What is this category.



This couldn't possibly be $D^b\mathrm{Coh}(X)$ for any X.

Why not?

(1) $D^b\mathrm{Coh}(X)$ has $\oplus$. Given any E, F $\exists$ obj. $E \oplus F$ s.t. $\hom(E \oplus F, -) \cong \hom(E, -) \times \hom(F, -)$

For this reason we consider a "completion" of $\mathrm{Fuk}(M)$. (When $M = pt$ and when $M =$ anything)

Philosophy: Any property that a math. object Y has the object $\mathrm{Maps}(X, Y)$ inherits (for any X).

The properties that $D^b\mathrm{Coh}(\sim)$ has (algeb. prop) are the properties that $\mathrm{Chain}_{\mathbb{Z}}$ has.

If we want $\mathrm{Fuk}(M)$ to have these as well, consider:

$\mathrm{Fun}_{A_\infty}(\mathrm{Fuk}(M)^{op}, \mathrm{Chain}_{\mathbb{Z}})$.

<u>Lemma</u> (Yoneda) $\exists$ a fully faithful embedding

$\mathrm{Fuk} \longrightarrow \mathrm{Fun}_{A_\infty}(\mathrm{Fuk}^{op}, \mathrm{Chain}_{\mathbb{Z}})$

<u>Rmk</u> Fully faithful embed. means the image of functor is a copy of $\mathrm{Fuk}(M)$ itself.

<u>Rmk</u> Holds for any A_∞ cat. $\mathcal{C}$ (not just $\mathcal{C} = \mathrm{Fuk}(M)$)

The functor: $\mathcal{C} \longrightarrow \mathrm{Fun}_{A_\infty}(\mathcal{C}^{op}, \mathrm{Chain}_{\mathbb{Z}})$

$L \longmapsto \hom_{\mathcal{C}}(-, L)$

<u>Rmk</u> A functor of A_∞ categories should consist of:

- a function $\mathrm{ob}\,\mathcal{C} \xrightarrow{F} \mathrm{ob}\,\mathcal{D}$
- a linear map $\hom_{\mathcal{C}}(L_0, L_1) \longrightarrow \hom_{\mathcal{D}}(F(L_0), F(L_1))$
- a bunch other maps saying the above data respects composition up to homotopy, up to homotopy, ...

The functor $\hom_{\mathcal{C}}(-, L) : \mathcal{C}^{op} \longrightarrow \mathrm{Chain}_{\mathbb{Z}}$

Given $X \in \mathrm{ob}\,\mathcal{C}$: $X \longmapsto \hom_{\mathcal{C}}(X, L)$

on morphisms $f : X_0 \to X_1$. Given $f \in \hom_{\mathcal{C}}(X_0, X_1)$ we obtain a map $\hom_{\mathcal{C}}(X_1, L) \longrightarrow \hom_{\mathcal{C}}(X_0, L)$

$g \longmapsto g \circ f$

called precompo. by f.