MATH 277: FUKAYA CATEGORIES, FALL 2015

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1. 2015-9-1

Based on the two people who have filled out the survey so far (fill it out!), it seems we are collectively uncomfortable with some basic objects that are relevant. So we will stick with the basics for the first half of today.

The first half of this course will give three good easy examples of mirror symmetry.

Let's start the math.

1.1. Basics.

Let M be a C^∞ manifold of dimension $2n$. A *symplectic form* on M is a choice of closed form $\omega \in \Omega^2_{\mathrm{dR}}(M, \mathbb{R})$ such that ω^n defines a volume form. A pair (M, ω) is called a *symplectic manifold*.

Remark. ω defines an isomorphism $\omega : \Gamma(TM) \to \Gamma(T^*M)$ given by applying ω to the tangent vector. This has an analogue in physics, where given a energy function H on a manifold, we may want to compute alternately dH or its associated vector field X_H.

Examples of symplectic manifolds include points, arguably the empty set depending on what we consider its dimension to be, and also nonsilly things. $\mathbb{R}^2$ with whatever nonzero constant choice of 2-form works, for instance. $M = \mathbb{R}^2 n$ has 2-forms $\sum_i dx_i \wedge dy_i$. Any orientable 2-manifold works for obvious reasons. Perhaps most interestingly, we have, for any smooth manifold Q, that the cotangent bundle T^*Q—on which we have the canonical 1-form (the Liouville form) given by $\theta(v)_{(q,p)\in T^*Q} = p(d\pi(v))$—has the symplectic form $d\theta$.

Exercise 1.1. Verify that $(T^*Q, d\theta)$ is a symplectic manifold.

A final example is a Kähler manifold, for which the imaginary part of the hermitian metric is symplectic.

Now, call a submanifold $L \subset M$ of the symplectic manifold M a *Lagrangian* if $\dim L = \frac{1}{2} \dim M$ and $\omega|_L = 0$. This object seems artificially defined to us at first glance, but we have the following nice theorem indicating a potential for deeper significance.

Theorem 1.2 (Gromov)**.** *We have the following.*

- *In $\mathbb{R}^{2n}$ with a usual symplectic structure, any compact Lagrangian has $H^1_{\mathrm{dR}}(L, \mathbb{R}) \neq 0$.*
- *There exist symplectic structures on $\mathbb{R}^{2n}$ such that $S^n \subset \mathbb{R}^2 n$ is a Lagrangian.*

An *almost complex structure* on a manifold M is a endomorphism J of TM such that $J^2 = -1$. J is called *compatible* with ω if $\omega(\bullet, J\bullet)$ is a Riemannian metric. A smooth map $u : (M, J) \to (M', J')$ is called (J, J')*-holomorphic* if $du \circ J = J' \circ du$. For instance, if these are complex manifolds, the J, J'-holomorphic maps are precisely the holomorphic maps.

1.2. **Toward Fukaya categories.** This will be nonrigorous! Feel the demand for rigor churn in your stomach, and deal with it somewhere else.

Fix Lagrangians $L_0, \dots$ of M, and assume L_i, L_j intersect simply transversally. We play a fun game. This game is called Morse theory. Set $P(L_i, L_j) = \{\gamma : [0, 1] \to M | \gamma(0) \in L_i, \gamma(1) \in L_j\}$.

Fix some component $P_0 \subseteq P$, fix a basepoint $\gamma_0 \in P_0$. Then for all paths $\gamma \in P_0$, we have some path from γ_0 to γ, that is, a function $u : [0, 1] \times [0, 1] \to M$, so we can take the number $A(\gamma) = \int_{[0,1]\times[0,1]} u^*\omega$. Then the *rate of change* of A is independent of γ_0 and of u itself (ω is closed!). We do Morse theory with dA. This will be our basic function. Because Morse theory needs it apparently, we also need a metric on P_0, but we can abduct it from a metric on M by integrating the length of the perturbation of the path over the path. So now we need a metric on M, which we get by choosing an almost complex structure J and setting $g = \omega(\bullet, J\bullet)$.

The following lemma is more difficult than it looks because infinite dimensional analysis is not easy.

Lemma 1.3 (Floer)**.** *$dA(\gamma) = 0$ iff γ is a constant path. So the set of critical points of A are in bijection with points $L_i \cap L_j$.*

The *gradient trajectories* are maps $u : \mathbb{R} \times [0, 1] \to M$ such that $u|_{\mathbb{R}\times\{0\}} \subset L_i$, $u|_{\mathbb{R}\times\{1\}} \subset L_j$, $\lim_{\to\infty} u(\bullet, s) = p \in L_i \cap L_j$, $\lim_{\to-\infty} u(\bullet, s) = q \in L_i \cap L_j$, and u is (i,J)-holomorphic treating $\mathbb{R} \times [0, 1]$ as a subset of $\mathbb{C}$.

2. 2015-9-4

We start with an exercise.

Exercise 2.1. Let $V \subset \mathbb{C}$ be open, and $u : V \to (M, \omega, J)$ be holomorphic (with the structures on M compatible). What does $\int_V u^*\omega = 0$ imply?

Fill out the survey! Before 23:59 on Tuesday, please. Otherwise, everything here will be at the wrong level, and you may have serious difficulties with this class. There will also be no class on Monday, in recognition of the struggles of the American labor movement.

So, apparently we don't all know Morse theory. So let's learn it.

2.1. **Morse theory.** Fix a smooth manifold X with riemannian metric and a smooth function $f : X \to \mathbb{R}$. One can define a graded abelian group out of the data (X, f). Then, using g we will make a differential, such that in good cases, we can recover $H^*(X)$ from the cohomology of the complex.

We define this graded abelian group to have generators $\mathrm{Crit}(f) = \{x \in X | df_x = 0\}$. There are some immediate issues, and we will ask that f is sufficiently generic that its critical points are discrete. We ask that it have the *Morse property* of the following lemma.

Lemma 2.2. *For all smooth manifolds X almost every C^∞ function f satisfies the* Morse property*: for every critical point of f, there is a coördinate chart about x such that $f = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_{\dim X}$.*

Remark. The number of negative signs above is coördinate-independent and is known as the index of f at x, denoted by $\mathrm{ind}_x(f)$.

With that out of the way, with f a Morse function, we let the graded abelian group be

$$\bigoplus_{x \in \mathrm{Crit}(f)} \mathbb{Z}[\mathrm{ind}(x)],$$

where that bracket puts the $\mathbb{Z}$ in degree ind(x).

Example 2.3. *If we let X be the usual 2-sphere and f be the usual height function, the graded abelian group has components $\mathbb{Z}$ in degree 0 and 2, and zero elsewhere.*

Now we find a differential. Define a putative differential ∂ sending the n graded piece to the $n-1$ graded piece by $\partial p = \sum_{|q|=|p|-1} n(p, q)q$, where $n(p, q)$ is the number of *discrete* gradient trajectories from q to p– counted with a sign gotten by considering these gradient trajectories to be the intersection of the ascending manifold from q (all points that end up at q eventually) with some orientation, and the descending manifold from p with some other rientation–where a gradient trajectory is a C^∞ map $\mathbb{R} \to X$ with tangent vector equal to ∇f everywhere. We know this is a discrete set because of our restriction of the index of p and q to be one away. In general, between critical points of index k and k' the path space of gradient trajectories modulo translation is just $k-k'-1$ (this is a theorem we invite you to look up at home!) A warning: this only sometimes works, and requires f be nice in other respects.

But when is ∂ a differential? The answer: not always! We need a bunch more assumptions to assure this: for instance, X must be compact. In this case, we first note that if I is some compact $1-manifold$, perhaps with boundary, ∂I occurs in pairs. If I is oriented, the *signed* count of ∂I is zero. So we want to realize the "broken trajectories" as the boundary of a 1-manifold. To do so, we take the moduli of paths p to r with $\text{ind}(r) = \text{ind}(p) - 2$, which is one-dimensional, and compactify such that the broken paths are its boundary. Of course, this only works if this one-manifold with boundary is proper, which in turn requires that the original manifold be compact.

3. 2015-9-9

Exercise 3.1. 1. Let M be a symplectic manifold with $\dim M \neq 0$. Show that if $\omega = d\theta$ for some $\theta \in \Omega^1$, then M must be non-compact or have boundary. An M admitting such a θ is called *exact*.
2. Let $H : M \to \mathbb{R}$ be a C^∞ function, and X_H the dual vector field to dH. Let $\Phi^H : M \times \mathbb{R} \to M$ denote the flow. Show that if $L \subset M$ is lagrangian, then so is $L \times \mathbb{R} \hookrightarrow (M \times T^*\mathbb{R}, \omega_M \oplus \omega_{T^*\mathbb{R}})$ given by $(x,t) \mapsto (\Phi^H(x,t), t, -H(\Phi^H(x,t)))$
3. Show that H is constant along the flow of X_H.

These should be pretty easy.

Now, last time we had a broad overview of Morse theory, where we constructed a chain complex based on the critical points of a Morse function f on a compact riemannian manifold and a nice associated differential on the complex. Upshot is that the homology of this Morse complex is just the ordinary singular homology.

Today, we will fantasize about "compactifying" 1-dimensional moduli spaces of holomorphic strips (polygons). That is, fixing $L_1, \ldots, L_n \subset M$ lagrangians intersecting transversally (where M has a compatible almost complex structure), we studied functions $u : \mathbb{R} \times [0,1] \to M$ satisfying (for fixed i, j):

(1) The boundary points $u(t,0) \in L_i$, $u(t,1) \in L_j$;
(2) $\lim_{t\to\pm\infty} u(t, \bullet) \subset L_i \cap L_j$;
(3) $J \circ du = du \circ i$.

Let's say that for some reason we can pick out a real 1-dimensional space of such strips u. What does a compactification of this one-dimensional space look like? One such limit point is two strips glued together at 0. Another possibility is the formation of an energy bubble on an interior point, or on the boundary. There is some confusion within the class about whether two bubbles can form at once, and we have some discussion about that with little resolution.

Critically, it seems like doing Morse theory as usual (defining a d so that $dp = \sum_q n_{p,q} q$, where $n_{p,q}$ is the number of holomorphic strips from q to p) results in $d^2 \neq 0$, being the sum of some of these boundary objects. But if we know that spheres and discs can't appear, we do have $d^2 = 0$. One such situation is the aforementioned *exact* manifolds, where ω is exact. And a lagrangian manifold L in M is called *exact* if there is some $f : L \to \mathbb{R}$ such that $\theta|_d f$. These conditions exclude the problematic boundary objects, as an embedded sphere would need to have positive area, whereas Stokes promises zero area.

Theorem 3.2. *There is a set up in which if we define* $CF^*(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}[p]$ *with* dp *as above, then* $d^2 = 0$ *assuming* M, L *exact and* spin*, where you should look up a spin lagrangian elsewhere.*

Alas, it looks like we won't define the Fukaya category today. We are out of time.

4. 2015-9-11

Exercise 4.1. 1. Let M be a closed manifold. Show that if any of the even dimensional de Rham cohomology groups vanish, M cannot be symplectic.
2. Let $M = T^*Q$, Q a smooth manifold. Given any $Z \subset Q$ a smooth manifold, define $T^*_Z(Q) := \{(z, \alpha) | z \in Z, \alpha \in T^*Q|_Z, \alpha|_{TZ} \equiv 0\}$. Show that is an exact lagrangian submanifold.
3. Fix (M, ω). Prove a compatible almost complex structure J exists, and that the space of compatible J is contractible.

Last time we motivated the choice of an exact symplectic manifold $(M, d\theta)$, looked at lagrangian submanifolds such that $\theta|_{L_i} = df_i$ for some f_i. We also looked at the space of deformations of paths from L_i to L_j.

Now a proofless, detail-lacking theorem.

Theorem 4.2. *For all such* L_i *we can define a cochain complex, the* Floer cochain complex*, generated by the transverse intersection points of* L_i *and* L_j*, where* d *describes holomorphic strips.*

If you are familiar with dg categories, this may look familiar, but we are not, so let's us define it.

Definition 4.3. Fix a ring k. A *dg category* $\mathcal{C}$ over k is the data of:

- A set of objects $\mathrm{ob}(\mathcal{C})$
- For all objects X, Y, a cochain complex of k-modules $\hom^*(X, Y)$
- A composition map $\hom^*(Y, Z) \otimes_k \hom^*(X, Y) \to \hom^*(X, Z)$ consistent with the chain complex structure on each side (meaning the natural total chain complex structure on the left-hand side).

This composition map must be associative and have a unit in $\hom^0(X, X)$ for all X.

Remark. Given a dg category $\mathcal{C}$, one can create its *homotopy category* $H^0(\mathcal{C})$ with the same objects as $\mathcal{C}$ and the arrows of $\hom^0(X, Y)$.

Example 4.4. *The category of k-cochain complexes, in which objects are cochain complexes and $\hom^i(X, Y)$ are degree i linear maps $X \to Y$ (i.e. a collection of maps sending X^i to Y^{j+i}), not necessarily chain maps. Then given $f \in \hom^i(X, Y)$, df is defined as sending x to $d_Y f(x) - (-1)^{i+1} f(d_X x)$.*

Remark. The zeroeth cohomology of $\hom^*(X, Y)$ is just cochain maps mod homotopy.

Now, the question of the day is whether we can define a dg category whose objects are L_i and with $\hom^*(L_i, L_j) = CF^*(L_i, L_j)$. Alas, we cannot (unless we loosen things os that that equality is just a homotopy). The structure that actually pops out is that of an A_∞ category. Exploring, what we want is a map $CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$ that sends points in the intersections of $L_1 \cap L_2$ and $L_0 \cap L_1$ to some combination of points of $L_0 \cap L_2$. To do so, we count holomorphic triangles with vertices at p, q, r with p, q, r in $L_1 \cap L_2$ etc.

Definition 4.5. Given p, q, r, define $n^r_{p,q}$ as the number of holomorphic triangles in M with boundary conditions, e.g. holomorphic maps from the closed unit disc minus three boundary points to M with boundary on L_0, L_1, L_2 such that the removed points would "sent" to p, q, r respectively.

Now define composition $\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$ by $\mu^2(p \otimes q) = \sum_{r \in L_0 \cap L_2} n^r_{p,q} r$. There are two things to check: that this is a map of complexes, and that it is associative. The first thing is okay, the second goes wrong.

5. 2015-9-14

Exercise 5.1. 1. Consider $\mathbb{R}^{2n}$ with its standard symplectic form. Let $\mathrm{GrLag}(\mathbb{R}^{2n})$ be the set of $V \subset \mathbb{R}^{2n}$ linear such that V is a Lagrangian. Show

$$\mathrm{GrLag}(\mathbb{R}^{2n}) \cong U(n)/O(n),$$

and compute the fundamental group.
2. Show a dg category with one object is the same thing as a differential graded algebra (unital, associative, not necessarily graded commutative) .
3.In an A_∞ category $\mathcal{C}$, suppose there is some threefold multiplication map $\mu^3 : \hom(L_2, L_3) \otimes \hom(L_1, L_2) \otimes \hom(L_0, L_1) \to \hom(L_0, L_3)[-1]$ such that $\mu^1\mu^3 + \mu^3(\mu^1, -, -) + \mu^3(-, \mu^1, -) + \mu^3(-, -, \mu^1) = \mu^2(\mu^2, -) + \mu^2(-, \mu^2)$. Show that $\hom_{H^*(\mathcal{C})}(X, Y) := H^* \hom_{\mathcal{C}}(X, Y)$ is a category enriched over graded abelian groups. This primarily entails showing that there is some associative composition law.

Tonight at 6:30 in Emerson 108 is a Harvard Gender Inclusivity in Mathematics kick-off event.

Last time: We defined $\mu^1 : \hom(L_i, L_j) \to \hom(L_i, L_j)[1]$ using holomorphic strips, and a multiplication map $\mu^2 : \hom(L_1, L_2) \otimes \hom(L_0, L_1) \to \hom(L_0, L_2)$. But this does not define a dg category. To prove associativity in μ^2, we might try to look at the boundary of a one dimensional component of disks that look like the complex disk punctured four times on the boundary mapping in the natural way: this has boundary consisting of the disk with a vertex bubbled off, or with two adjacent points bubbled off.

Remark. This degeneracy list is very similar to the boundary list for strips, since the two are conformally equivalent modulo two additional boundary points.

To rephrase, let $\mathcal{M}_{x_0,x_1,x_2,y}$ be the set of u sending the relevant boundaries to L_0, L_1, L_2, L_3 in a way compatible with the almost complex structure. Let $\mathcal{M}^1$ be the one-dimensional components. There is a compactification, which is a one-dimensional manifold with boundary. Algebraically, we have

$$0 = \mu^3(\mu^1 x_2, x_1, x_0) + \ldots + \mu^2(\mu^2(x_2, x_1), x_0) + \mu^2(x_2, \mu^2(x_1, x_0)) + \mu^1\mu^3(x_2, x_1, x_0),$$

where μ^3 is defined in terms of strips going from three points x_0, x_1, x_2 to y. Hey! This is the exercise from earlier!

In particular, associativity may not hold, unless μ^3 vanishes in the way we want. This essentially never happens, to the extent that we are not willing to hypothesize that it does. Instead, μ^2 is only associative up to homotopy (given by μ^3). But μ^3 was a choice. We want to show that composing ≥ 4 elements has coherent associativity.

Definition 5.2. Let $\mu^k : \hom(L_{k-1}, L_k)\otimes\cdots\otimes\hom(L_0, L_1) \to \hom(L_0, L_k)[2-k]$ be the set of y in $L_0 \cap L_k$ weighted by disks going from $x_0, \dots, x_k$ to y.

Theorem 5.3. *These operations satisfy the A_∞ relations:*

$$0 = \sum_{\substack{u+t+r=k \\ s=r+u+1}} \mu^s(1^{\otimes r} \otimes \mu^t \otimes 1^{\otimes u}).$$

And we have that an A_∞ *category* os the data of a collection of objects $\mathcal{C}$ with graded hom k modules and a multilinear map μ^k as above satisfying the A_∞ relations.

6. 2015-9-16

Exercise 6.1. Consider the Clifford torus T^n in $\mathbb{CP}^n$. Show that this torus is lagrangian, show that $D_i = [1 : \dots : 1 : z : 1 : \dots : 1]$ with $|z| < 1$ a holomorphic disk in $\mathbb{CP}^n$ with boundary on the Clifford torus. Show that $T\mathbb{CP}^n|_{D_i} \cong D_i \times \mathbb{C}^n$ and the tangent space to the Clifford torus gives a map $S^1 = \partial D_i \to \mathrm{GrLag}(\mathbb{C}^n)$. Final question: what element of $\pi_1(\mathrm{GrLag}(\mathbb{C}^n))$ does this represent?

This becomes more relevant once we have the additional context of the *Maslov index*.

Talk topics have been posted on Piazza. Get 'em while they're hot.

Last time we saw the idea of why Lagrangians in an exact symplectic manifold M, counting holomorphic polygons with boundaries in L_i should give rise to the structure of an A_∞ category. We still have to figure out some important things, like the grading of this category, and the transversality question of why holomorphic maps to M satisfying suitable boundary conditions constitute a smooth manifold. Relatedly, what happens if the intersections of Lagrangians are not transverse. There are gluing questions, like how we compactify $\mathcal{M}$, the relevant moduli space, and signs questions, about how we orient $\mathcal{M}$. Assume these issues can be resolved for now. Time for a bad definition!

Definition 6.2 (Not actually correct, but flavorful)**.** Fix M. The Fukaya category of M has objects the Lagrangians and hom sets given by the $CF^\bullet$ with A_∞ multiplication operations μ^k as given last time.

Remark. Last time we did not mention units of an A_∞ category, that is, $\mathrm{id}_L \in \hom(L, L)$. So definition from last time is of a non-unital category. Our quick fix is to say an A_∞ category $\mathcal{C}$ if $H^\bullet\mathcal{C}$ is unital.

6.1. **A baby form of homological mirror symmetry.**

Conjecture (Kontsevich 1994 ICM)**.** *For all Calabi-Yau manifolds X—those where the canonical bundle is trivial—there is a Calabi-Yau manifold $X^\vee$ and equivalences of A_∞ categories $D^\pi Fuk(X) \simeq D^b\mathrm{Coh}(X^\vee)$ and $D^b\mathrm{Coh}(X) \simeq D^\pi Fuk(X^\vee)$, where D^π is some weird closure operator that your correspondent doesn't understand yet.*

Now, there are a lot of directions we can go in from here for e.g. talks. One is understanding $CF^*(L, L)$ (requires analysis, Morse theory,...). Another is symplectic homology, which is pretty big. There are a bunch of others, posted online.

7. 2015-9-18

You should fill out the talk survey, and start posting what solutions you have online. Like for srs.

7.1. **Continued work with the to-do list.** Smoothness of $\mathcal{M}$, the moduli of holomorphic maps from the disk to M satisfying a particular boundary condition. The answer? It's not. Sorry.

To set up, fix the unit disk $D \subset \mathbb{C}$ and puncture $k+1$ boundary points $x_0, \ldots, x_k$. Set $\hat{D} = D\backslash\{x_i\}$. Different choices of $\{x_i\}$ give rise to non-equivalent complex manifolds in general, as the automorphisms of the unit disk only are triply transitive on the boundary. So, there is instead a moduli space of choices of $k+1$ points on the boundary of D modulo biholomorphic automorphism. Let $\tilde{\mathcal{R}}$ be the moduli of $k+1$ unordered points on S^1, and let $\mathcal{R}$ be the same space modulo PSL_2. It's worth noting that this is a manifold for $k \geq 2$. It is not "stacky or orbifoldy in any way". For instance, $\mathcal{R}_{2+1}$ is a point. Now, when $k \geq 3$ we let $\mathcal{M}$ consist of pairs (u, S) where S is a choice of holomorphic structure on D minus $k+1$ points and u is a smooth map to M. And there is a bundle $\mathcal{E}$ over $\mathcal{M}$ whose fiber at a point is the set of all smooth sections of u^*TM. And there is a second vector bundle $\mathcal{F}$ over $\mathcal{M}$ with fiber $\hom(TS, u^*TM)$. Now, we have the operation $\overline{\partial}u = \frac{1}{2}(du + Jduj)$, which is naturally 0 for holomorphic u. This produces a section $\overline{\partial}$ of $\mathcal{F}$, which is zero precisely when u is holomorphic. We hope that this subset should be a C^∞ manifold if the graph of $\overline{\partial}$. But it almost never is a transverse intersection, so we will perturb the $\overline{\partial}$ equation and or change the fibers of $\mathcal{F}$. To address the first of those, we choose a 1 form on S with values in $C^\infty(M)$, e.g., $Y \in \Omega^1_{\mathrm{dR}}(S, C^\infty(M))$, and we perturb by studying the differential equation $\overline{\partial}(u) = X_Y + JX_Yj$, where X_Y is the natural map $TS \to \Gamma(TM)$ given by Y put together with the symplectic form. The way we change $\mathcal{F}$ is looking at the subbundle (respecting some almost complex structure (?)?). Then there's some choice of perturbations producing a transverse intersection.

8. 2015-9-21

Last time, starting with the setup of an exact symplectic manifold M, we examined the set $\mathcal{R}_{k+1}$ of holomorphic structures on D^2 minus $k+1$ points, or equivalently the number of conformal structures of the disk with that many punctures. It has dimension $k-2$ by standard considerations. We then look at the space of pairs (u, S) with S a holomorphic structure and u a map $S \to M$ a map sending the boundary to chosen lagrangian submanifolds and satisfying the perturbation-of-holomorphic condition from last time.

Remark. This also resolves the issue of lagrangians not intersecting transversally, by focusing attention away from the intersection itself, describing it instead as a *hamiltonian chord*.

Definition 8.1. A *hamiltonian chord* is a C^∞ map $c : [0, 1] \to M$ such that $c(0) \in L_{i-1}$, $c(1) \in L_i$, and $c'(t) = X_Y(c(t))$ for the perturbing hamiltonian Y.

Theorem 8.2. *There is a pair of choices of J_x, Y making the moduli space of pairs (u, S) a C^∞ manifold, where J_x is a compatible almost complex structure on M for all x in S.*

Today, we ask what the dimension of the space of pairs u, S is. The idea is that given a map $u : D^2 \to M$ sending the tangent space of the boundary is sent to $\mathrm{LagGr}(M)$, and we can do a dimension count via this in some way. In particular, the winding number of this map determines the dimension of the space, and is called the *Maslov index* of u.

Definition 8.3. A symplectic manifold M is *almost Calabi-Yau* if $c_1(TM) = 0$.

Given a almost Calabi-Yau manifold, after choosing a trivialization, letting $L = \mathrm{LagGr}(M)$, there is a map $L \to S^1$. If this map lifts to some α, then α is called a *grading* on L. Fun fact, the ability to shift a grading by an arbitrary integer echoes the ability to shift the L complex as an object of the Fukaya category.

We are now prepared to talk about the graing on $CF^*(L_0, L_1)$. Suppose the lagrangians have gradings α_0 and α_1. Then the degree of $x \in L_0 \cap L_1$ is the winding number of a path from $\alpha_1(x)$ to $\alpha_0(x)$ ending with negative derivative at $\alpha_0(x)$, taking the convention that winding numbers are determined by the signed number of passes with $\alpha_0(x) + \mathbb{Z}$.

9. 2015-9-23

Last time, we started thinking about the dimension of $\mathcal{M} = \{(u, S)\}$ and started assigning gradings to the intersections of graded lagrangians. The goal of today is to sketch an argument for the following claim.

Claim 9.1. *Fix graded lagrangians L_i, α_i $0 \leq i \leq k$, and consider the moduli $\mathcal{M}$ of pairs (u, S) of maps from a holomorphic disk S to M satisfying the usual boundary conditions with particular points $x_i \in L_{i-1} \cap L_i$ being the limits as we approach punctures. Then the dimension of $\mathcal{M}$ is the index of the incoming point x_0 minus the index of all the outgoing points plus $k - 2$.*

We remind ourselves that the *index* of a point indicates its degree in the CF complex.

Remark. This whole nonsense explains why μ^k is a degree $2 - k$ map, as that allows the relevant moduli space to have dimension 0.

First, we will examine strips $S = \mathbb{R} \times [0, 1]$, and fix a pseudoholomorphic map u satisfying the boundary conditions. Now, last time we claimed that you compute $\dim \mathcal{M}_u$ by examining some winding number $u|_{\partial s} : \partial s \to \mathrm{LagGr}$.

Now, how do we make a loop in LagGr? We assume the maps $\mathbb{R} \times \{x\} \to \mathrm{LagGr}(\mathbb{C}^n)$ are compactly supported, which determines ins some convoluted way your correspondent doesn't understand a loop in the lagrangian grassmannian. A theorem due to Floer says that the winding number of the path duly produced is the dimension of $\mathcal{M}_u$. To prove this, Floer does some serious real analysis and computes some nonsense called the *spectral flow*, which he shows is equal to both of what we want.

Of course, the obvious question is how this relates then to $i(x_0)$ and $i(x_1)$. Essentially, the answer is that it works, regardless, of course, of our choice of grading. To generalize, we can glue disks using a holomorphic connected sum and consider $\overline{\partial}$ operators.

Example 9.2. *Let M be the complex numbers with the usual symplectic structure and the usual almost holomorphic structure. Pick a trivialization of the squared determinant of the tangent bundle. Let L_0 be a line, and L_1 a weird curve that intersects it twice. The claim is that there is a unique holomorphic strip between the two, up to dull automorphisms. This is a consequence of Riemann mapping.*

10. 2015-9-25

Today, examples!

10.0.1. *Background.* Given an exact symplectic manifold $(M, \omega = d\theta)$, can we find $M^\vee$ such that the Fukaya category of M is the *dg* enhancement of $D^b\mathrm{Coh}(M^\vee)$.

10.0.2. *Example.* Consider M a point. Its tangent space is rank zero, but its determinant bundle is still $\mathbb{R}$. Then $\omega = 0$ is a symplectic form, and the only corresponding lagrangian other than the empty set is a single point. For a grading, a choice of integer suffices (when L is a point; when it's the empty set there's nothing to grade). Based on what we've discussed so far, an object of $\mathrm{Fuk}(M)$ is a pair (L, α) where L is a lagrangian and α a grading, so there's a $\mathbb{Z}$ worth of objects in this category plus the empty object. To compute hom spaces in the category, we first note that the empty set is irrelevant (all hom sets involving it will be empty). Otherwise, we should have $CF^*(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z} = \mathbb{Z}$ with grading $n' - n$, where n, n' are the gradings of L_0, L_1 respectively.

Now, let's compute μ^1, μ^2 etc. μ^1 must be zero for two reasons, one, because our graded abelian group has no other choice (having only one $\mathbb{Z}$ factor, in one single degree), and second, because a holomorphic map u from the strip must be constant, so $\mathcal{M}_u/\mathbb{R}$ has dimension -1. μ^2 is really a map $\mathbb{Z}[-(n_2 - n_1)] \otimes \mathbb{Z}[-(n_1 - n_0)] \to \mathbb{Z}[-(n_2 - n_0)]$. We claim this is the multiplication map. For there is a unique punctured disk map to M for any triple of points in intersections of lagrangians, so the product of two generators maps to a generator.

Finally, we claim $\mu^k \equiv 0$ for $k \geq 3$. For then the choice of holomorphic structure is not fixed in a u, S pair, and all points of the moduli are hence contained in dimension not zero. So multiplication is zero.

Then, what is the category? Well, we've described it precisely... But the upshot is that this category cannot be $D^b\mathrm{Coh}(X)$ for any X, as for instance direct sums would exist otherwise. For this reason, we consider a "completion" of the Fukaya category in general. The general philosophy is that any property that an object Y has is inherited by the objects $\mathrm{Maps}(X, Y)$. This comes up in that the properties that $D^b\mathrm{Coh}(X)$ has are the algebraic properties that chains have. Hence, to let the Fukaya category have these algebraic properties as well, consider the functor space $\mathcal{F}un_{A_\infty}(Fuk(M)^{\mathrm{op}}, Chain_{\mathbb{Z}})$.

Lemma 10.1 (Yoneda's lemma)**.** *There is a fully faithful embedding* $Fuk \to \mathcal{F}un_{A_\infty}(Fuk(M)^{\mathrm{op}}, Chain_{\mathbb{Z}})$.

Remark. This result is not quite as stupid as the first Yoneda lemma (what is an A_∞ functor, anyway?), but is not hard and applies to general A_∞ categories. The map is just like in the original Yoneda lemma.

11. 2015-9-28

Last time, we looked at homological mirror symmetry for our manifold a point, and noticed that the Fukaya category was not a dg enhancement of a derived category of bounded complexes of coherent sheaves of any variety, so we realized we needed to modify our notion of mirror symmetry to make it work out well.

Today, we set up and/or introduce basic dg category tools and terminology to define the so-called $D^\pi Fuk(pt)$ category, also known as the Karoubi completion of $Fuk(pt)$ or Perf(Fuk(pt)). We claim $D^\pi Fuk(pt) \simeq D^b\mathrm{Coh}(\mathrm{Spec}\mathbb{Z})$ as A^∞ categories. Then tensoring with $\mathbb{C}$ produces good mirror symmetry.

Fun fact!. Originally, mirror symmetry was not formulated over $\mathbb{C}$ or $\mathbb{Z}$, but instead over the so-called *Novikov ring*, but under certain circumstances we can use these nicer rings instead.

By Yoneda's lemma there is a fully faithful embedding $Fuk(pt) \hookrightarrow \mathrm{Fun}_{A_\infty}(Fuk(pt)^{\mathrm{op}}, \mathrm{Chain}_\mathbb{Z})$ sending X to $\hom_{Fuk(pt)}(\bullet, X)$. And by the philosophy from last time, the big category on the right has all the nice algebraic properties that $\mathrm{Chain}_\mathbb{Z}$ has (e.g. it inherits a dg structure from $\mathrm{Chain}_\mathbb{Z}$). In addition, there's a coproduct of two objects (which is also a product!), and hence finite coproducts. In addition, mapping cones exist; for all morphisms $f : A \to B$, there is an object $\mathrm{Cone}(f) \in \mathrm{Chain}_\mathbb{Z}$ and a commutative (up to homotopy) diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & \mathrm{Cone}(f), \end{tikzcd}$$

and such that all other commutative diagrams of this sort admits a unique map from the cone. In the chain complex category, in fact, we can show that $\mathrm{Cone}(f) \simeq B \oplus A[1]$ with a well-chosen differential that can be found in any homological algebra textbook.

In addition, we have in our category that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & D, \end{tikzcd}$$

realizes D as a mapping cone if and only if it realizes A as a *homotopy kernel* of g, in the same sense that the mapping cone is the homotopy cokernel. Neither of these are the things you get if you only think of kernels and cokernels of chain complexes, without homotopy considerations.

Finally, all idempotents—maps $f : A \to A$ such that $f \circ f = f$—there exists an object R and maps i, r such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "r"] & A \\ R \arrow[u, "i"] \arrow[r, "id"] & D, \arrow[u, "i"] \end{tikzcd}$$

commutes up to homotopy. We claim that $\text{Chain}_{\mathbb{Z}}$ gives all these properties to $\text{Fun}_{A_\infty}(Fuk(pt)^{\text{op}}, \text{Chain}_{\mathbb{Z}})$, essentially defining everything pointwise in an ad-hoc manner.

So now there is an obvious candidate for $D^\pi Fuk(pt)$, the smallest full subcategory of $\text{Fun}_{A_\infty}(Fuk(pt)^{\text{op}}, \text{Chain}_{\mathbb{Z}})$ containing the image of the Yoneda embedding and satisfying the conditions we talked about above. This is called the *Karoubi completion*.

Remark. If X is smooth and projective, the dg enhancement of $D^b\text{Coh}(X)$ also satisfies the properties listed above.

We now have enough stuff to prove the first claim of the day. We stsart by noting that the image of the Yoneda embedding of $Fuk(pt)$ is just $\{0, \mathbb{Z}, \mathbb{Z}[\pm 1], \ldots\}$. Since any $F \in \text{Fun}_{A_\infty}$ is the colimit of representables (that is, of functors of the form $\hom(\bullet, Y)$), there is a functor $\text{Chain}_{\mathbb{Z}} \to \text{Fun}(Fuk, \text{Chain}_{\mathbb{Z}})$, the smallest full subset of $\text{Chain}_{\mathbb{Z}}$ containing the image and satisfying the properties above is the category of bounded degree finitely generated complexes, as was to be shown.

Remark. For X affine, smooth, noetherian, we have

$$D^b\text{Coh}(X) \simeq D^b(\text{fg}H^0(\mathcal{O}_X)\text{−Mod}).$$