Math 277 Lecture ~~#~~ 11 9/28/2015

①

<u>Last time:</u> stated studying HMS for $M = pt$

$$\mathrm{Fuk}(pt) = \quad \phi \quad \begin{array}{l} \xrightarrow{0} pt[1] \circlearrowleft \mathbb{Z} \\ \rightleftarrows\ pt[0] \circlearrowleft \mathbb{Z} \quad \mathbb{Z}[1] \\ pt[-1] \circlearrowleft \mathbb{Z} \end{array}$$

$\Rightarrow$ $\nexists$ variety X s.t. $\mathrm{Fuk}(pt) = D^b\mathrm{Coh}(X) \leftarrow$ dg category.

because $\mathrm{Fuk}(pt)$ ~~doesn't~~ doesn't even have $\oplus$, but $D^b\mathrm{Coh}(X)$ ~~possesses~~ has $\oplus$.

<u>Today:</u> Set up ~~[illegible]~~ basic dg category terms to define $D^\pi\mathrm{Fuk}(pt)$

(Karoubi completion of $\mathrm{Fuk}(pt)$) (aka $\mathrm{Perf}(\mathrm{Fuk}(pt))$)

Claim: $D^\pi\mathrm{Fuk}(pt) \cong D^b\mathrm{Coh}(pt)$

$\uparrow$ A_∞-category $\cong D^b\,\mathbb{Z}\text{-Mod}^{f.g.} \hookleftarrow$ ~~this~~ is the variety $\mathrm{Spec}\,\mathbb{Z}/\mathbb{Z}$.

Novikov ring
Novikov field ?

$(\Rightarrow D^\pi\mathrm{Fuk}(pt) \otimes_{\mathbb{Z}} \mathbb{C} \simeq D^b\mathrm{Coh}(pt) \hookleftarrow \mathrm{Spec}\,\mathbb{C}/\mathbb{C}\ )$

<u>Idea:</u> By Yoneda's lemma, $\exists$ fully faithful embedding $\mathrm{Fuk}(pt) \hookrightarrow \mathrm{Fun}_{A_\infty}(\mathrm{Fuk}(pt)^{op}, \mathrm{Chain}_{\mathbb{Z}})$

$X \longmapsto \hom_{\mathrm{Fuk}(pt)}(-, X)$

can be made in A_∞-category (Lyubaschenko)

By philosophy of last time, $\mathrm{Fun}_{A_\infty}(-, \mathrm{Chain}_{\mathbb{Z}})$ has all the nice props that $\mathrm{Chain}_{\mathbb{Z}}$ has.

Rmk: in general, $\mathrm{Fun}_{A_\infty}(\mathcal{C}, \mathcal{D})$ is an A_∞-category. If $\mathcal{D}$ is a dg category, so is $\mathrm{Fun}_{A_\infty}(\mathcal{C}, \mathcal{D})$

<u>The properties we like:</u>

(1) $\exists \oplus$, given $A, B \in \mathrm{Chain}_{\mathbb{Z}}$, $\exists!\, A \oplus B$ (up to equivalence) s.t. $\forall C \in \mathrm{Chain}_{\mathbb{Z}}$.

$$\hom_{\mathrm{Chain}_{\mathbb{Z}}}(A \oplus B, C) \simeq \hom_{\mathrm{Chain}_{\mathbb{Z}}}(A, C) \times \hom_{\mathrm{Chain}_{\mathbb{Z}}}(B, C)$$

equivalence of chain complexes

$$\simeq \hom_{\mathrm{Chain}_{\mathbb{Z}}}(A, C) \oplus \hom_{\mathrm{Chain}_{\mathbb{Z}}}(B, C)$$

induced by $A \hookrightarrow A \oplus B$, $B \hookrightarrow A \oplus B$

(i.e. $\mathrm{Chain}_{\mathbb{Z}}$ admits finite coproducts)

$A \oplus B$ is called the coproduct of A and B.

can check: $A \oplus B \simeq B \oplus A$ $\quad \oplus$ associate (by equivalences above).

(2) Mapping cones exist.

$\forall f: A \to B$, $\exists$ $\mathrm{Cone}(f) \in \mathrm{Chain}_{\mathbb{Z}}$ and a commutative up to homotopy diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "H"] & B \arrow[d] \\ 0 \arrow[r] & \mathrm{Cone}(f) \end{tikzcd}$$

~~which~~ where commutes up to ~~co~~ hommotopy means

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "H"] & B \arrow[d, "g"] \\ C \arrow[r, "i"'] & D \end{tikzcd}$$

we have specified ~~les~~ $H \in \hom^{-1}(A, D)$ s.t. $dH + Hd = gf - ih$

s.t. $\forall$ othere diagrams
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g'"] \\ 0 \arrow[r] & D' \end{tikzcd}$$
, $\exists!$ (up to contractible choice)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "g"] \arrow[ddr, "g'"] & \\ 0 \arrow[r] \arrow[drr] & \mathrm{Cone}(f) \arrow[dr] & \\ & & D' \end{tikzcd}$$

~~Commutes~~.

Explicitly, we can prove $\mathrm{Cone}(f) \simeq B \oplus A[1]$ with d_B, f, $d_A(-1)$; $\simeq$

$$\begin{array}{c} d_B \circlearrowleft B^1 \oplus A^2 \circlearrowright -d_A \\ \nwarrow f \\ B^0 \oplus A^1 \\ d_D \circlearrowleft \nwarrow f \circlearrowright d_A \\ B^{-1} \oplus A^0 \end{array}$$

(3).
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & \cancel{\mathrm{Cone}(f)}\, D \end{tikzcd}$$
realizes ~~Cone(f)~~ D is a mapping cone iff it realizes A as a homotopy kernel of g

Rmk: Unless f is injective, $\mathrm{Cone}(f) \not\simeq \mathrm{Coker}(f)$ $\leftarrow$ take naively $\mathrm{Coker}(f_i: A_i \to B_i) = B_i/\mathrm{im} f_i$

Ex: $A \xrightarrow{0} 0$ ($f = 0$, $B = 0$), then $\mathrm{coker}(f) = 0$. $\mathrm{Cone}(f) = A[1]$

(4) Idempotents split (i.e. $\exists$ "image" for idempotents) $\forall f: A \to A$. and $f \circ f \overset{H}{\sim} f$,

$\exists R$, i, r.
$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[dr, "r"] & A \\ R \arrow[ur, "i"] \arrow[rr, equal, "\mathrm{id}_R"'] & & R \arrow[u] \end{tikzcd}$$
commuting (up to homotopy)

i.e. left kan ??

Since $\mathrm{Chain}_{\mathbb{Z}}$ has these properties, so does $\mathrm{Fun}_{A_\infty}(\mathrm{Fuk}(pt), \mathrm{Chain}_{\mathbb{Z}})$

why? given $A \xrightarrow{f} B$ in Fun_{A_∞}, $\forall X \in \mathrm{Fuk}(pt)$, let $C(X) = \mathrm{Cone}(A(X) \xrightarrow{f_X} B(X))$

then C is a mapping cone for f. (Define things pointwise)

So now there's an obvious candidate for $\overline{D^\pi \mathrm{Fuk}(pt)}$:

We're going to use a standard method in analysis (Grothendieck used it) to define $D^\pi \mathrm{Fuk}(pt)$ i.e taking closure of image.

Def. Given $\mathrm{Fuk}(pt) \longrightarrow \mathrm{Fun}_{A_\infty}(\mathrm{Fuk}^{op}, \mathrm{Chain}_{\mathbb{Z}})$

define $D^\pi \mathrm{Fuk}(pt) \subset \mathrm{Fun}_{A_\infty}(\mathrm{Fuk}^{op}, \mathrm{Chain}_{\mathbb{Z}})$ to be the smallest subcategory.

- containing image (Yoneda embedding)
- satisfying (1) ~ (4)

Rmk: If X is smooth projective, $D^b\mathrm{Coh}(X)$ (dg enhancement) satisfies (1)–(4).

Claim: $\mathrm{Perf}(\mathrm{Fuk}(pt)) \simeq D^b\mathrm{Coh}(pt)$
$\quad\quad = D^\pi \mathrm{Fuk}(pt)$

Pf (Sketch):

Note any functor of the form $\hom_{\mathrm{Fuk}}(-, Y)$ is determined completely by the value on $pt \in \mathrm{Fuk}(pt)$.

So $\exists$ an equivalence.

$$\begin{array}{ccc} \text{image (Yoneda embedding)} & \subset & \mathrm{Fun}_{A_\infty} \\ \wr\vert & & \downarrow \mathrm{res}\vert_{pt} \\ \{0, \mathbb{Z}, \mathbb{Z}[\pm 1], \dots\} & \subset & \mathrm{Chain}_{\mathbb{Z}} \end{array}$$

Since any $F \in \mathrm{Fun}_{A_\infty}$ is a ~~[illegible]~~ colimit of representables (i.e. a colimit of functors of the form $\hom(-, Y)$ $Y \in \mathrm{ob}\,\mathrm{Fuk}$), $\exists$ a functor

$$\mathrm{Chain}_{\mathbb{Z}} \xrightarrow{\text{full, f}} \mathrm{Fun}(\mathrm{Fuk}, \mathrm{Chain}_{\mathbb{Z}})$$

the smallest full subcategory of $\mathrm{Chain}_{\mathbb{Z}}$ containing $\{0, \mathbb{Z}, \dots\}$ (in particular $\mathbb{Z}$)

[given $\mathbb{Z}$ $(\mathbb{Z} \xrightarrow{id} \mathbb{Z}) = 0$. $\mathrm{Cone}(\mathbb{Z} \to 0) = \mathbb{Z}[1]$ ...]

satisfying (1)–(4), is the category of cochain complexes bounded in degree + finitely generated in each degree.

Rmk: For X ~~[illegible]~~ affine, smooth, Noeth.

$$D^b\mathrm{Coh}(X) \simeq D^\pi \text{f.g } R\text{-Mod} \quad R = \Gamma(\mathcal{O}_X)$$

Next time: examples in $T^*\mathbb{R}$ and in T^*S^1

$$W\,\mathrm{Fuk}(T^*S^1) \simeq D^b\mathrm{Coh}(\underbrace{\mathbb{A}^1 \setminus pt}_{\mathbb{C}^\times}) \quad \Leftarrow \text{Eventual goal.}$$