

Last time: stated studying HMS for  $M=pt$

$$Fuk(pt) = \begin{array}{ccc} & \xrightarrow{\circ} & pt \in \mathbb{Z} \circ \mathbb{Z} \\ \phi & \xleftrightarrow{\circ} & pt \in [0] \circ \mathbb{Z} \\ & \xrightarrow{\circ} & pt \in [-1] \circ \mathbb{Z} \end{array} \left. \vphantom{\begin{array}{ccc} & \xrightarrow{\circ} & pt \in \mathbb{Z} \circ \mathbb{Z} \\ \phi & \xleftrightarrow{\circ} & pt \in [0] \circ \mathbb{Z} \\ & \xrightarrow{\circ} & pt \in [-1] \circ \mathbb{Z} \end{array}} \right\} \mathbb{Z}[1]$$

$\Rightarrow \nexists$  variety  $X$  st  $Fuk(pt) = D^b Coh(X) \leftarrow$  dg category.

because  $Fuk(pt)$  ~~doesn't~~ doesn't even have  $\oplus$ , but  $D^b Coh(X)$  ~~has~~ has  $\oplus$ .

Today: Set up ~~basic~~ basic dg category terms to define  $D^\pi Fuk(pt)$

(Karoubi completion of  $Fuk(pt)$ ) (aka  $Perf(Fuk(pt))$ )

Claim:  $D^\pi Fuk(pt) \cong D^b Coh(pt)$   
 $\uparrow$   
 $A_{\infty}$ -category  $\cong D^b \mathbb{Z}\text{-Mod}^{fg} \hookrightarrow$  this is the variety  $Spec \mathbb{C}/\mathbb{Z}$ .

Novikov ring  
 Novikov field

( $\Rightarrow D^\pi Fuk(pt) \otimes_{\mathbb{Z}} \mathbb{C} \cong D^b Coh(pt) \hookrightarrow Spec \mathbb{C}/\mathbb{C}$ )

Idea: By Yoneda's lemma,  $\exists$  fully faithful embedding  $Fuk(pt) \hookrightarrow \text{Fun}_{A_{\infty}}(Fuk(pt)^{op}, \text{Chain}_{\mathbb{Z}})$   
 $x \mapsto \text{hom}_{Fuk(pt)}(-, x)$   
 can be made in  $A_{\infty}$ -category (Lyubaschenko)

By philosophy of last time,  $\text{Fun}_{A_{\infty}}(-, \text{Chain}_{\mathbb{Z}})$  has all the nice props that  $\text{Chain}_{\mathbb{Z}}$  has.

Amk: in general,  $\text{Fun}_{A_{\infty}}(\mathcal{C}, \mathcal{D})$  is an  $A_{\infty}$ -category. If  $\mathcal{D}$  is a dg category, so is  $\text{Fun}_{A_{\infty}}(\mathcal{C}, \mathcal{D})$

The properties we like:

(1)  $\exists \oplus$ , given  $A, B \in \text{Chain}_{\mathbb{Z}}$ ,  $\exists! A \oplus B$  (up to equivalence) st  $\forall C \in \text{Chain}_{\mathbb{Z}}$ .

$$\text{hom}_{\text{Chain}_{\mathbb{Z}}}(A \oplus B, C) \cong \text{hom}_{\text{Chain}_{\mathbb{Z}}}(A, C) \oplus \text{hom}_{\text{Chain}_{\mathbb{Z}}}(B, C)$$

$\uparrow$  equivalence of chain complexes

$$\cong \text{hom}_{\text{Chain}_{\mathbb{Z}}}(A, C) \oplus \text{hom}_{\text{Chain}_{\mathbb{Z}}}(B, C)$$

induced by  $A \hookrightarrow A \oplus B$ ,  $B \hookrightarrow A \oplus B$

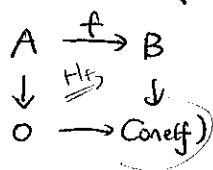
c.i.e Chain $_{\mathbb{Z}}$  admits finite coproducts

$A \oplus B$  is called the coproduct of  $A$  and  $B$ .

can check:  $A \oplus B \cong B \oplus A$   $\oplus$  associate (by equivalences above).

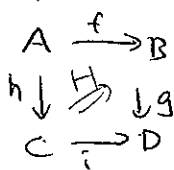
(2) Mapping cones exist.

$\forall f: A \rightarrow B, \exists \text{ Cone}(f) \in \text{Chain}_{\mathbb{Z}}$  and a commutative up to homotopy diagram.



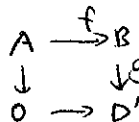
which where

commutes up to homotopy means

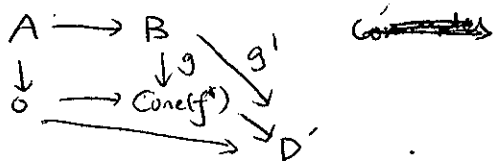


we have specified  $H \in \text{hom}^1(C, D)$  s.t.  $dH + Hd = gf - ih$

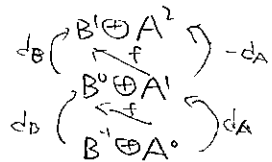
st.  $\forall$  others diagrams.



$\exists!$  (up to contractible choice)



Explicitly, we can prove  $\text{Cone}(f) \simeq B \oplus A[1]$

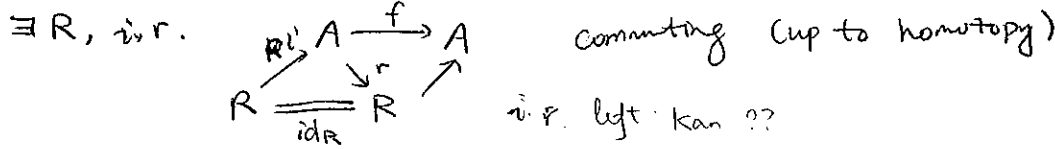


(3)  $A \xrightarrow{f} B$  realizes  $\text{Cone}(f)$  is a mapping cone iff it realizes  $A$  as a homotopy kernel of  $g$

Rmk: Unless  $f$  is injective,  $\text{Cone}(f) \neq \text{Coker}(f)$  ← take naively  
 $\text{Coker}(f: A_i \rightarrow B_i) = B_i / \text{im } f_i$

Ex:  $A \xrightarrow{\begin{smallmatrix} + \\ 0 \\ 0 \end{smallmatrix}} B$ , then  $\text{coker}(f) = 0$ ,  $\text{Cone}(f) = A[1]$

(4) Idempotents split (i.e.  $\exists$  "image" for idempotents)  $\forall f: A \rightarrow A$  and  $f \circ f \sim f$



Since  $\text{Chain}_{\mathbb{Z}}$  has these properties, so does  $\text{Fun}_{\text{Ab}}(\text{Fuk}(pt), \text{Chain}_{\mathbb{Z}})$

why? given  $A \xrightarrow{f} B$  in  $\text{Fun}_{\text{Ab}}^{\text{F}}$ ,  $\forall X \in \text{Fuk}(pt)$ , let  $C(X) = \text{Cone}(A(X) \xrightarrow{f_X} B(X))$   
then  $C$  is a mapping cone for  $f$ . (Define things pointwise)

So now there an obvious candidate for "D<sup>h</sup> Fuk(pt)"

We're going to use a standard method in analysis (Grothendieck used it) to define  $D^h \text{Fuk}(pt)$  i.e taking closure of image.

Def: Given  $\text{Fuk}(pt) \rightarrow \text{Fun}_{\text{Ab}}(\text{Fuk}^{op}, \text{Chain}_{\mathbb{Z}})$

define  $D^h \text{Fuk}(pt) \subset \text{Fun}_{\text{Ab}}(\text{Fuk}^{op}, \text{Chain}_{\mathbb{Z}})$  to be the smallest subcategory.

- containing image (Yoneda embedding)
- satisfying (1) ~ (4)



