9/30.

Exer. Let $\mathcal{C} = Ab$ be the category of all abelian groups. Let $\mathcal{C}^0 = \{0\}$ the full subcategory of only 0. Let $\mathcal{C}' = \{\mathbb{Z}\}$ the full subcat whose only object is $\mathbb{Z}$.
Compute the smallest full subcat of $\mathcal{C}$ containing $\mathcal{C}'$ closed under $\oplus$, kernels, cokernels, and idempotents.

Rmk. Not a dg-category problem. You know what (co)kernels, $\oplus$ and idempotents splittings are for abelian groups.

Bonus. R comm. ring. Do the same exercise when $\mathcal{C} = R\text{-mod}$, $\mathcal{C}^0 = \{0\}$, $\mathcal{C}' = \{R\}$.

<u>Note</u>: In Δ'd cat, an extension

$A \to E \to B$ is a cokernel

$$\begin{array}{ccccc} A & \to & E & & \\ \downarrow & & \downarrow & & \\ 0 & \to & B & \to & A[1] \\ & & \downarrow & & \downarrow \\ & & 0 & \to & E[1] \end{array}$$

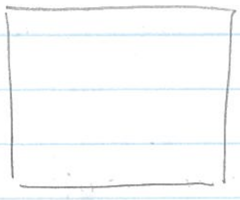
If $f \circ f = f$ ($\Leftrightarrow f \circ f \overset{H^2}{\sim} f$
if $\hom^{-1}(X, Y) = 0$)

$$f \circ (f \circ f) \overset{H^3}{\sim} (f \circ f) \circ f$$

C^∞ mfld   scheme$/\mathbb{Z}$.

Last time: $D^\pi \mathrm{Fuk}(pt) \simeq D^b \mathrm{Coh}(pt)$

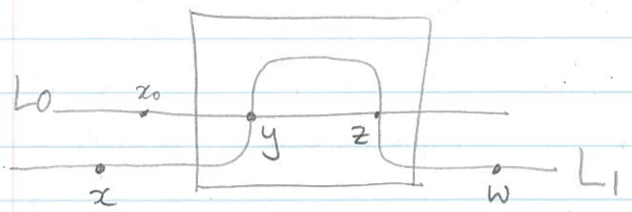
Today: higher dim'l examples.

Ex. 

$M = \mathbb{C}$

$\omega = dx\,dy$

J = mult. by i.



Thm/Lemma. (Automatic regularity) No need to perturb the equation $\bar\partial = 0$.

(If M is 2-dim'l$/\mathbb{R}$, $\{L_i\}$ are in general position, no need to alter Ψ, J)

$L_i \pitchfork L_j$, no triple intersections.

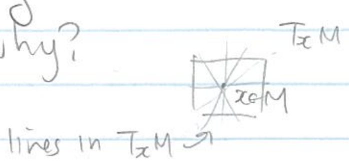
$CF^*(L_1, L_0) = ?$

Need to choose a grading $\alpha_i : L_i \to \mathbb{R}$ for each i.

$TM \cong M \times \mathbb{C}$

$\mathrm{GrLag}(TM) \cong M \times \mathbb{RP}^1$

why?



The $\det^2$ map:

$$\begin{tikzcd} M \times \mathbb{RP}^1 \arrow[r, "\det^2"] \arrow[d, "\cong"'] & S^1 \ni e^{2i\theta} \\ M \times \frac{U(1)}{O(1)} \arrow[ur] & \end{tikzcd}$$

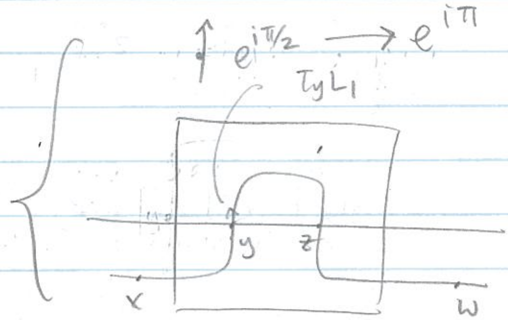
$x \in [0,1)$

$$
\begin{tikzcd}
 & \mathrm{GrLag}(M) \cong M \times \mathbb{RP}^1 \arrow[r, "\det^2"] \arrow[d] & S^1 \\
L_0 \arrow[ur, "T_x L_0"] \arrow[r] \arrow[urr, bend right, "x_0 \in L_0 \mapsto 1 = e^{i0}"'] & M &
\end{tikzcd}
$$

constant.

$$
\begin{tikzcd}
 & \mathbb{R} \arrow[d] \\
L_0 \arrow[ur, "\alpha_0"] \arrow[r, "\det^2"'] & S^1
\end{tikzcd}
\quad \longleftrightarrow \quad \alpha_0(x_0) \in \mathbb{Z}.
$$

$$
\begin{aligned}
L_1 &\longrightarrow S^1 \\
x &\longmapsto 1 = e^{i0} \\
y &\longmapsto e^{i\pi} \\
z &\longmapsto e^{i\pi} \\
w &\longmapsto 1 = e^{i0}
\end{aligned}
$$



Given α_1 a grading,

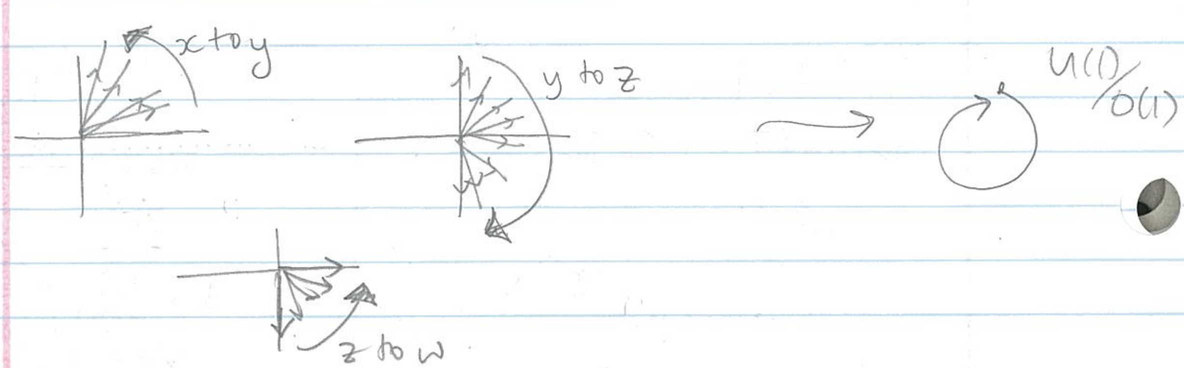
$$
\longleftrightarrow \quad
\begin{tikzcd}
 & \mathbb{R} \arrow[d, "e^{2\pi i t}"] \\
L_1 \arrow[ur, "\alpha_1"] \arrow[r] & S^1
\end{tikzcd}
$$

$\alpha(x) = n$

$\alpha(y) = n + \frac{1}{2}$

$\alpha(z) = n + \frac{1}{2} - 1 = n - \frac{1}{2}$

$\alpha(w) = n$



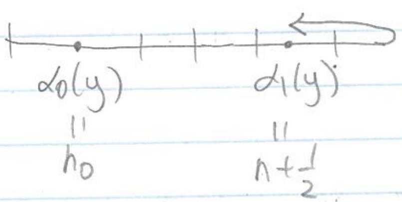
Give integers to $y, z \in L_0 \cap L_1$

$CF^*(L_1, L_0) = \mathbb{Z}[-|y|] \oplus \mathbb{Z}[-|z|]$.

$$\begin{array}{ll} \vdots & \\ \uparrow & \\ \mathbb{Z} & |z| \\ \uparrow d=? & \\ \mathbb{Z} & |y|. \\ \uparrow & \\ \vdots & \end{array}$$

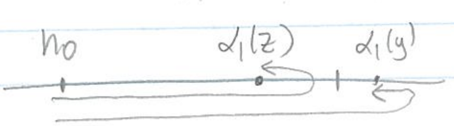
__Claim__ $|y| = |z| + 1$.

To compute $|y|$, we look at a path $\tilde{\gamma}$ in $\mathbb{R}$ from $\alpha_0(y)$ to $\alpha_1(y)$ w/ negative derivative @ $\alpha_1(y)$. and count # of crossings with $\alpha_1(y) + \mathbb{Z}$.



(should be $\lfloor \alpha_1(y) - \alpha_0(y) \rfloor$ or $\lfloor \alpha_0(y) - \alpha_1(y) \rfloor$)

Then we see $|y| = |z| + 1$.



Need to find holom strips $u: \mathbb{R} \times [0,1] \to M$

s.t. $\mathbb{R} \times \{0\} \to L_0$

$\mathbb{R} \times \{1\} \to L_1$

then $\partial z = \#\{u\}_{/\mathbb{R}}\, y$

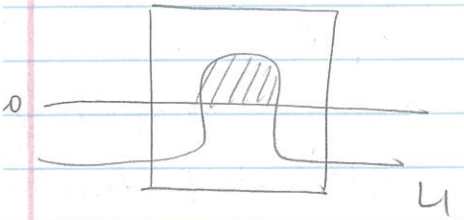
$\lim_{t \to +\infty} u = z$

$\lim_{t \to -\infty} u = y$

Thm (Riemann Mapping Thm w/ Boundary Conditions)
$\exists$ a strip satisfying these conditions

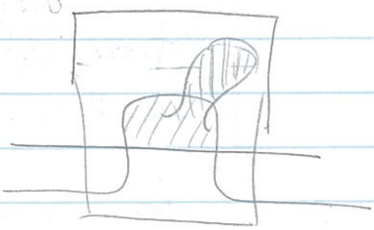
$\mathbb{R}\times[0,1]$ $\qquad$ $D^2\setminus\{z,z'\}$

Claim: This is the unique strip.



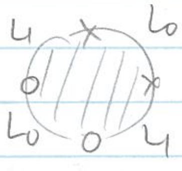
(by maximum principle, u must have obvious image:

not allowed:



???

• Doesn't fit ∂-cond^n on Lagrangians:



$\boxed{\text{So } \partial z = y}$ up to sign.

CF^*:
$$\begin{array}{ccc} y & \mathbb{Z} & |y| \\ \uparrow & \uparrow \partial & \\ z & \mathbb{Z} & |z|. \end{array}$$

$\Rightarrow H^* CF = 0.$

$$H^{|y|} = \ker(d^{|y|}) / \operatorname{im}(d^{|y|-1}) = \mathbb{Z}/\mathbb{Z} = 0$$

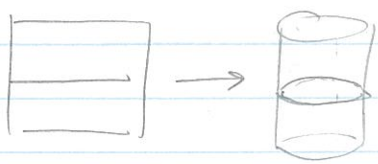
$$H^{|z|} = 0.$$

$(\mathbb{C}, dx \wedge dy, i)$

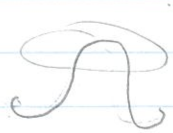
Consider: $\left(T^*\mathbb{R} = \text{—}\!\!+\!\!\text{—}\!\to x \ (y),\ dy \wedge dx, -i\right) / \mathbb{Z}.$

not the same but equivalent.

$$T^*\mathbb{R} \xrightarrow{/\mathbb{Z}} T^*S^1$$



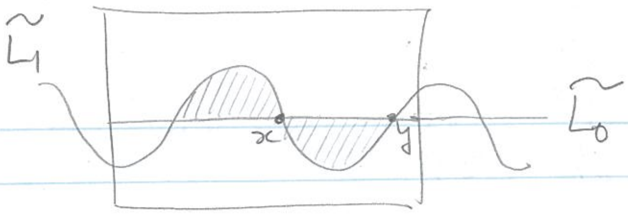
Let's compute  $= L_0 = S^1$

 $= L_1 =$ graph of df;
for $f: S^1 \to \mathbb{R}$ a Morse ftn
w/ two critical points.

$CF^*(L_0, L_1) = ?$

$$\begin{array}{ccccc} \mathrm{LagGr}(T^*S^1) & \to & S^1 & \ni & e^{2\pi i \theta} \\ & & & & \| \\ \uparrow & & \uparrow & & \uparrow \\ L_0 & \longrightarrow & \mathbb{R} & \ni & 0 \end{array}$$

Claim Give L_0 the zero grading.
Then one can give L_1 a grading
s.t. $|x| = \text{index}(x)$, $x \in L_0 \cap L_1 = \mathrm{Crit}(f)$



Say $|y| = |x| + 1$

(L_1, L_0)?

$\rightsquigarrow H^*(CF^\bullet(L_0, L_1)) \cong H^* L_0$

on: $\partial x = \underset{\substack{\uparrow \\ \text{sign?}}}{y - y} = 0$

Thm. If $L_1 \overset{\text{Hamiltonian isotopic}}{\sim} L_0$ exact setting,

then $HF^*(L_0, L_1) \cong H^* L_0$.

$\rightsquigarrow$ implies some conjecture