1 October 2.

Exercise 1.1. Let $f : M \to \mathbb{R}$ be Morse, and fix a generic metric g. Compare the Morse complex of f with that of $-f$. What can you say? (Specifically, what can you get about Poincaré Duality?)

Exercise 1.2. Let $f_i : M_i \to \mathbb{R}$ be Morse for $i = 1, 2$. What can you say about the Morse complex of $f_1 \times f_2 : M_1 \times M_2 \to \mathbb{R}$? (Specifically, what can you get about the Kunneth formula?)

Last time, we argued that $H^\bullet(CF^\bullet_{T\times S^1}(S^1, \Phi_t(S^1))) \cong H^\bullet(S^1)$, where Φ is a Hamiltonian. Today:

Theorem 1.1 (PSS isomorphism)**.** *Let $L \subset M$ be an exact Lagrangian, and assume it can be graded. Then there is an isomorphism*

$$HF^\bullet(L, \varphi(L)) \cong HM_{n-\bullet}(L) \cong H_{n-\bullet}(L).$$

Motivation: observe that vector fields that come from functions tend to have more zeros than arbitrary vector fields. For example, let $X = S^3$. This has all kinds of nowhere vanishing vector fields on it, being a Lie group. But assume $V \in \Gamma(TM)$ is equal to ∇f for some $f : S^3 \to \mathbb{R}$. By Morse theory, we know a generic function f must have at least two critical points, because $H_\bullet(S^3) = k \oplus k \cong HM_\bullet(S^3, f)$, so $CM_\bullet$ must have rank at least 2. (Alternatively, a function on a compact domain must have a maximum and a minimum.) In general for M compact, a generic f should give ∇f with at least $\sum_{i=0}^{\dim M} b^i(M)$ zeros.

Conjecture 1.1 (Arnol'd)**.** *Let $H : M \to \mathbb{R}$ be a generic Hamiltonian and $L \subset M$ a compact Lagrangian. Then $\#(L \cap \Phi_1^{X_H}(L)) \geq \sum b^i(L)$.*

The PSS isomorphism is a very special case of the Arnol'd conjecture.

Note that the ring structure on $H^\bullet(L)$ is much less information than the ring structure on $\Omega^\bullet(L)$. For example, it's a theorem (called a formality theorem) when $H^\bullet(X) \cong \Omega^\bullet(X)$ as a commutative differential graded algebra. This happens when X is a Lie group, S^n, or Kähler; such manifolds are called "formal manifolds".

Idea of PSS construction. Standing assumption: $CF^\bullet(L, \varphi^H(L))$ is equal to $CF^\bullet(L, L)$ where the pseudoholomorphic $(\cdot)^{0,1} = 0$ equation is perturbed compatibly with H. Idea: let $u : \mathbb{R} \times [0, 1] \to M$ with $(du - X_Y)^{0,1} = 0$. Near $\pm\infty$, u converges to a Hamiltonian chord along H_t. So a generator for $CF^\bullet(L, L) = CF^\bullet(L, \varphi^H(L))$ is the same thing as a time-1 chord from L to itself.

How do we get a map to $CM_{n-\bullet}$? Fix $f : L \to \mathbb{R}$, g a metric on L. Let $q \in Crit(f)$. We study the following moduli space: let $\gamma : \mathbb{R} \to L$ be such that $-\infty \mapsto q$ and $\dot\gamma(t) = -\nabla f(\gamma(t))$. Let $u : \mathbb{R} \times [0, 1] \to M$ be a holomorphic strip attached to (pinched at) $\gamma(0)$ on one end and having x at the other satisfying "some conditions".

Let $M(x, q)$ be the moduli space of $\{(\gamma, u)\}$. Define the PSS map to be

$$\Phi : CF^k(L, L) \to CM_{n-k}(L, f)$$

$$x \mapsto \sum_q \#M_{[0]}(x,q)q.$$

There's a map in the other direction

$$\Psi : CM_{n-k} \to CF^k$$

$$q \mapsto \sum \#M(q,x)x.$$

For this to induce a map on $H^\bullet$, we need to check that $d\Phi = \Phi d$ and $d\Psi = \Psi d$. It turns out that one can compactify the one-dimensional component of $M(x,q)$. From $q, \gamma, \gamma(0), u, x$ we get possible boundary elements that look like $q, \tilde{\gamma}, q', \tilde{\gamma}', \tilde{\gamma}(0), u, x$ and $q, \gamma, \gamma(0), x', x$. Our assumptions prevent bubbles popping out either on the strip or on the gradient trajectory. Thus

$$0 = \#(\partial \overline{M}_{[1]}(x,q)) = d_{Morse}\Phi \pm \Phi d_{Floer}.$$

The same proof works for Ψ.

How do we see this is an isomorphism on cohomology? Consider $\Phi \circ \Psi : CM_{n-k} \to CF^k \to CM_{n-k}$. The picture is a gradient trajectory from q to a point, expanding into a holomorphic strip at x, followed by another attached holomorphic strip going to another point, which has a gradient trajectory to q'. What PSS does (see Albers) is exhibit a cobordism from the 0-manifold counting such pictures to the 0-manifold counting gradient trajectories from q to q'. Note that in either case, the only possible γ, u are the constant ones. So $\Phi \circ \Psi = \mathrm{id}$.

Now consider $\Psi \circ \Phi : CF^k \to CM_{n-k} \to CF^k$. Now we're counting pictures consisting of a holomorphic strip from x' to a point, then a gradient trajectory to q, then a gradient trajectory to another point, then a holomorphic strip to x. We can construct a cobordism from the 0-manifold counting this to a 0-manifold counting holomorphic strips between x' and x. The way to do this is to change the relevant PDEs so that the gradient trajectories shrink to the point q, then cut off the strips near q and glue them together. Since $|x| = |x'|$, there are only constant strips. □

Some notes/details:

—How do we actually define $M(x,q)$? It is the pullback of the map $ev_0 : G(q) \times M(x) \to L \times L$, $(\gamma, u) \to (\gamma(0), u(-\infty))$, to L under the map $\Delta : L \to L \times L$, where

$$G(q) = \{\gamma : \mathbb{R} \to M \mid \gamma(-\infty) = q, \dot{\gamma} = -\nabla f\}$$

$$M'(x) = \{u : \mathbb{R} \times [0,1] \to M \text{ such that...}\}.$$

Here $(du - \beta X_Y)^{0,1} = 0$, where β is a function on $\mathbb{R} \times [0,1]$ such that $\beta \equiv 1$ on $t \gg 0$ and $\beta \equiv 0$ on $t \ll 0$, so $u(-\infty) \in L$, $u(+\infty) = x$, $u|_{\mathbb{R} \times \{0,1\}} \subset L$.