Hiro 10/2

Ex: $f : (M,g) \to \mathbb{R}$ Morse. $\rightsquigarrow$ Morse complex

What can you say if $f \rightsquigarrow -f$? (Poincaré duality)

Ex: $f_i : M_i \to \mathbb{R}$ $i = 1, 2$. $M_1 \times M_2 \xrightarrow{f_1 \times f_2} \mathbb{R}$? (Künneth)

Last time:

Argued that $H^*(CF^*_{T^*S^1}(S^1, \Phi_t(S^1))) \simeq H^*(S^1)$.

Today:

Thm (PSS isom.)

$L \subset M$ exact Lag., graded.

Then $HF^*(L, \phi(L)) \overset{\exists}{\simeq} HM_{n-*}(L) \; (\simeq H_{n-*}(L).)$

ϕ: Hamiltonian; HM: Morse homology.

Motivation:

Obs: v.f. that comes from fcn tend to have more zeros than arbitrary v.f.

e.g. $X = S^3$. (has many nowhere vanishing v.f.)

Assume $V \in \Gamma(TX)$ is equal to ∇f for some $f : S^3 \to \mathbb{R}$.

But by Morse th., a generic f must have at least 2 critical pts. In general, for M cpt, a generic f should yields ∇f with $\geq \sum_{i=0}^{n} b^i(M)$ zeros.

Conj (Arnold).

$H : M \to \mathbb{R}$ generic Hamiltonian, $L \subset M$ cpt. Lag.

Then $\#\{L \cap \Phi_1^{X_H}(L)\} \geq \sum b^i(L)$.

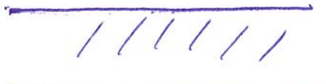
Obs: Ring str. on $H^*(L)$ has much LESS information than the ring of $\Omega^\bullet(L)$.

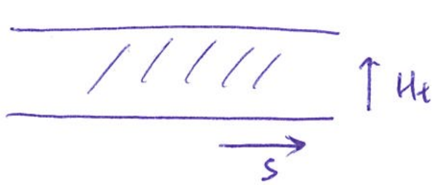
Formality thm. When $H^*(X) \simeq \Omega^\bullet(X)$. as a commu. dg alg.
(X Lie gps. S^n. Kähler).

Idea of construction. (PSS Born.):

Standing assumption:

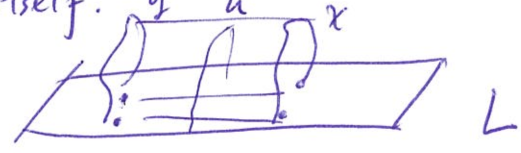
Note that $CF^*(L, \phi^H(L))$ is (very believably) equal to $CF^*(L,L)$. where $(\quad)^{0,1} = 0$ eq'n is perturbed compatible with H.


$\xrightarrow{u} M \qquad (du - X_H)^{0,1} = 0.$



near $s = \pm\infty$, u converges to a Ham. chord along H_t.

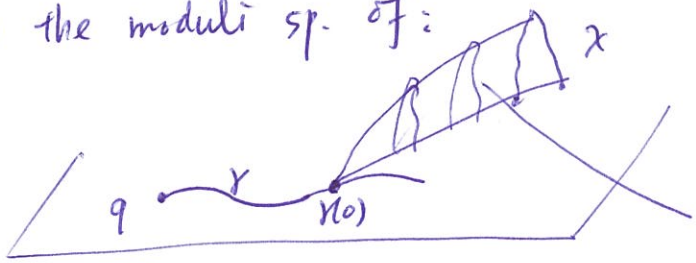
So a generator for $CF^*(L,L) = CF^*(L, \phi^H(L))$ is the same thing as a time-1 chord from L to itself.



How do we get a map to CM_{n-*} ?

Fix $f: L \to \mathbb{R}$, g: metric on L, $q \in \mathrm{Crit}(f)$.

Study the moduli sp. of:



$u: \mathbb{R} \times [0,1] \to M$

satisfying ...

$\gamma: \mathbb{R} \to L$

$-\infty \mapsto q$.

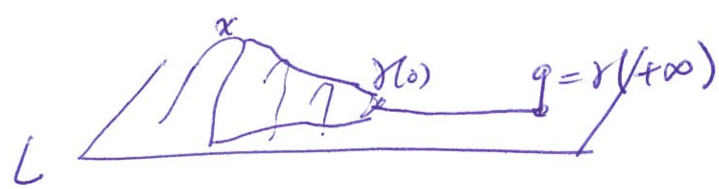
$\dot{\gamma}(t) = -\nabla f(\gamma(t))$.

Roughly, let $\mathcal{M}(x,q)$ be the moduli sp. of $\{(\gamma, u)\}$.

Define the PSS map by:

$$CF^k(L,L) \xrightarrow{\Phi} CM_{n-k}(L,f).$$
$$x \longmapsto \sum_q \# \mathcal{M}_{[0]}(x,q) \cdot q.$$

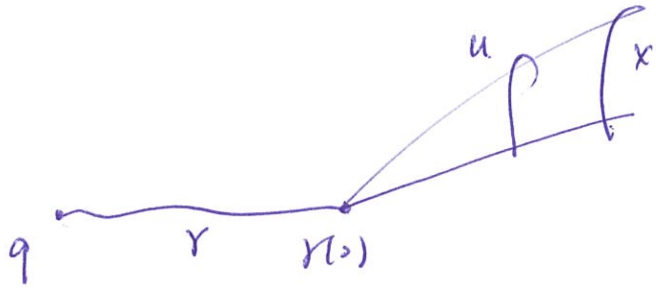
$$CM_{n-k} \xrightarrow{\Psi} CF^k$$
$$q \longmapsto \sum \# \mathcal{M}(q,x)\, x.$$



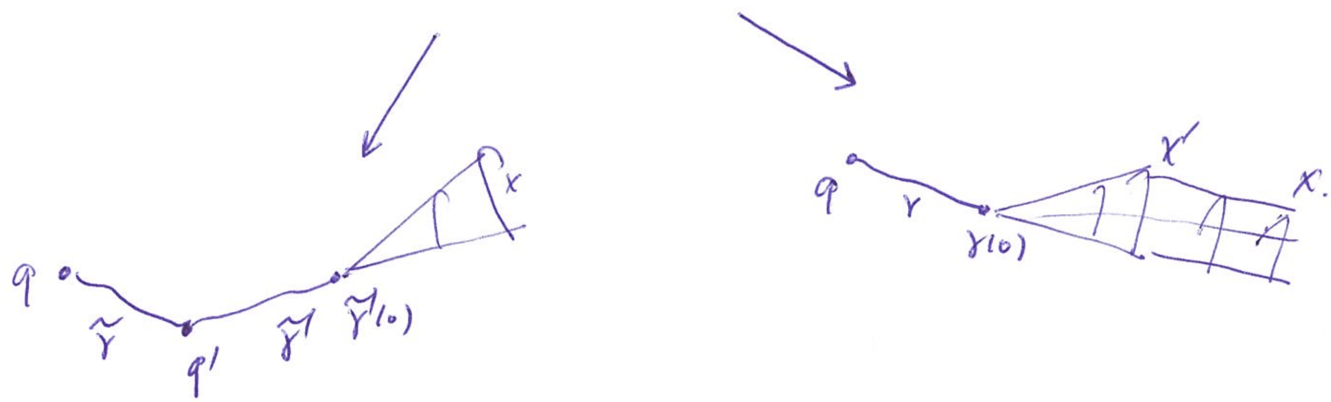
For this to induce a map on cohomology, need to check

$d\Phi = \Phi d$, $d\Psi = \Psi d$.

It turns out that one can compactify the 1-dim. comp. of $\mathcal{M}(x,q)$.



→ bubbles are excluded by exactness.



$$0 = \#\left(\partial \overline{\mathcal{M}}_{[1]}(x,q)\right)$$

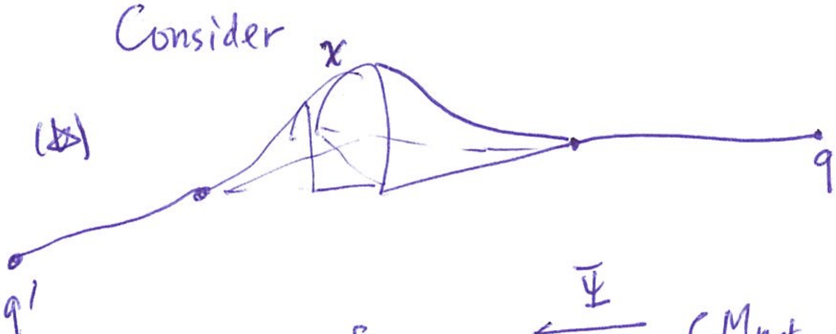
$$= d_{Morse}\,\Phi \pm \Phi\, d_{Floer}.$$

(Same proof for Ψ)

How to see this is an isom. on cohomology?

Consider

(⊛)



$$CM_{n-k} \xleftarrow{\Phi} CF_k \xleftarrow{\Psi} CM_{n-k}$$

What PSS does (See Albers) is exhibit a cobordism from the 0-mfd counting such picture (⊛) to the 0-mfd counting gradient trajectories from q to q'.

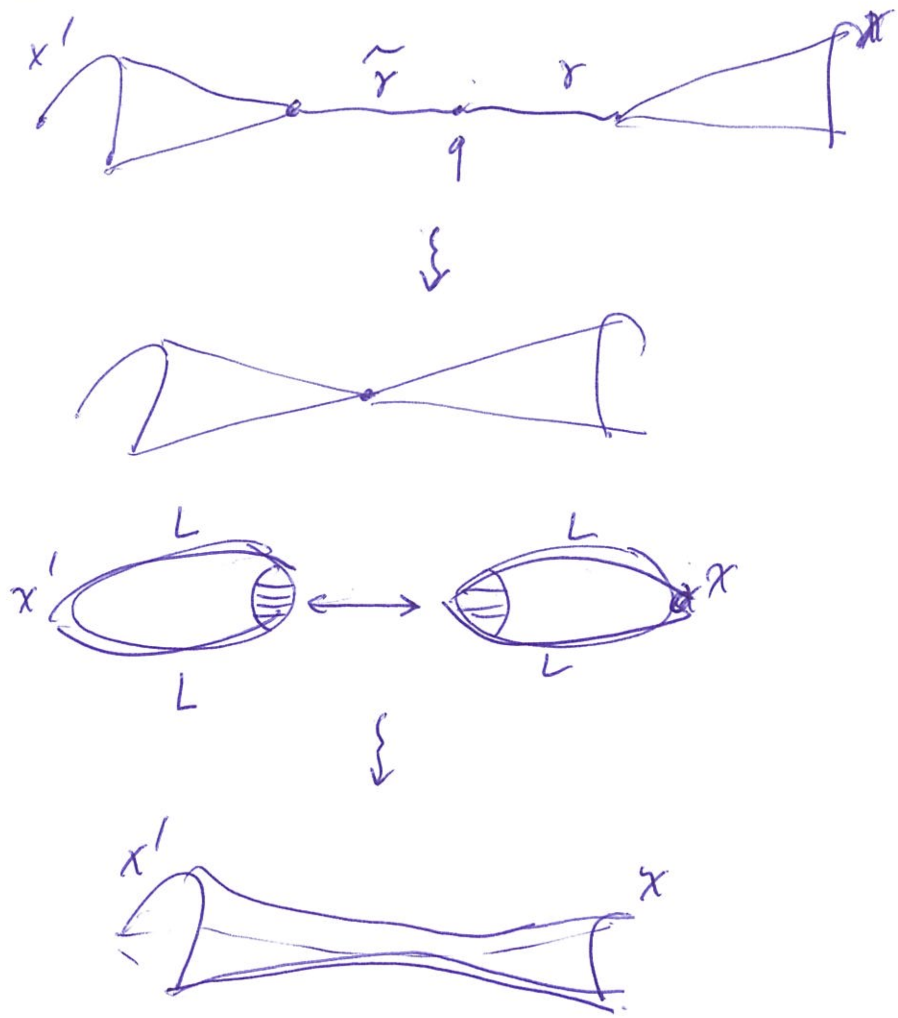
<u>Note:</u> In either case, the only possible γ, u are the constant ones.

So $\Phi \circ \Psi = \mathrm{id}$.

Now consider $CF^k \xleftarrow[\Psi]{} CM_{n-k} \xleftarrow[\Phi]{} CF^k$.

- Can also construct cobordism from the 0-mfd counting ( ). to a 0-mfd counting

Since $|x| = |x'|$, only constant strips. □



Some notes:

How do we actually define $\mathcal{M}(x,q)$?

$$\begin{tikzcd}
\mathcal{M}(x,q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G(q) \times \mathcal{M}'(x) \arrow[d] & (\gamma, u) \arrow[d, mapsto] \\
L \arrow[r, "\Delta"] & L \times L & (\gamma(0), \partial u(-\infty))
\end{tikzcd}$$

Where $G(q) := \{ \gamma : \mathbb{R} \to M \mid \gamma(-\infty) = q, \ \dot{\gamma} = -\nabla f \}$

$\mathcal{M}'(x) := \{ u : \mathbb{R} \times [0,1] \to M \mid (du - \beta X_y)^{0,1} = 0.$ where β is a fcn on $\mathbb{R} \times [0,1]$ s.t. $\beta \equiv 1$ on $t \gg 0$. $\beta \equiv 0$ on $t \ll 0$.

$u(-\infty) \in L$, $u(+\infty) = x$,

$u|_{\mathbb{R} \times \{0,1\}} \subset L$. $\}$