Kevin Sackel Fukaya Categories — L15 7 Oct 2015

- **Exercise**: Let $\mathcal{A}$ be a k-linear category with one object, $*$. Set $A := \mathrm{End}_{\mathcal{A}}(*)$. Show that a functor

  (1) $\mathcal{A} \longrightarrow k\text{-}\underline{\mathrm{Mod}}$ is the same as a left A-module

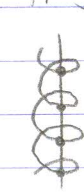
  (2) $\mathcal{A}^{op} \longrightarrow k\text{-}\underline{\mathrm{Mod}}$ is the same as a right A-module

- **Last Time**: Sketched:
  - $WCF^*(L,L)$
  - When $M = T^*S^1$, $\Theta = p\,dq$, $L = T_q^*S^1 \cong \mathbb{R}$

    $WCF^*(L,L) \cong H^*(WCF^*(L,L))$
  - $H^*(WCF^*(L,L)) \cong \mathbb{C}[t,t^{-1}]$ with $|t| = 0$

    $\mathbb{Z}[t,t^{-1}] \cong H^*(WCF^*(L,L))$ ↳ over $\mathbb{C}$, not $\mathbb{Z}$

Wrapping by Hamiltonian



- **Rmk**: No PSS isomorphism because L is non-compact.

$$\begin{matrix} CF^k(L,L) & \longrightarrow & CM_{n-k}(L) \\ H^{\cdot}\Big\{ & & \Big\} H^{\cdot} \text{ when } L \text{ is } \underline{\text{compact}} \\ H^k(L,L) & \longrightarrow & H^{sing}_{n-k}(L) \end{matrix}$$

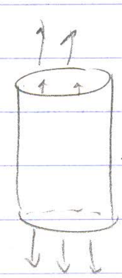
- **Note**: WCF looks like

$$D^{\pi}(WFuk(T^*S^1)) \xleftarrow{\exists\text{ embedding}} fg\ \mathrm{Mod}_{\mathbb{C}[t,t^{-1}]}$$

$$L = T_q^*S^1 \circlearrowleft \mathbb{C}[t,t^{-1}] \qquad\qquad X = \mathbb{C}[t,t^{-1}] \circlearrowleft \mathbb{C}[t,t^{-1}]$$

- **Mock defn**:

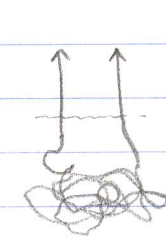
  ob $WFuk(T^*S^1)$ = exact Lagrangians $L \subset T^*S^1$ (with spin structure)



  s.t. $L \setminus K \cong \partial L \times \mathbb{R}_{>0}$

  K: some compact set; ∂L: some codim 1 submfld of L; $\mathbb{R}_{>0}$: flow parametrization by $X_{p\,dq}$

  Maybe, e.g., L looks like



- Note: More details later about μ^k operations.
  But the cochain complex $(CF(L_0, L_1), \mu^1)$ is as last time
  (generated by $L_0 \pitchfork \Phi_1^H(L_1)$ where H generates wrapping,
  and μ^1 counts holo^c strips)

- Thm [Abouzaid, weak version]:
  Let Q be a spin mfld. Then the object $T_q^* Q$, some $q \in Q$,
  split-generates the category $D^\pi(\mathrm{WFuk}(T^*Q))$,
  i.e any object in $D^\pi(\sim)$ can be obtained by successively
  taking finitely many $\oplus$, cones, and splitting idempotents starting with T_q^*Q.

- Cor: $D^\pi(\mathrm{WFuk}(T^*S^1)) \simeq D^b\mathrm{Coh}(\underbrace{\mathbb{A}^1_{\mathbb{C}} \setminus pt}_{= \mathrm{Spec}\, \mathbb{C}[t,t^{-1}]})$

  $\simeq D^b \, \mathrm{fgMod}_{\mathbb{C}[t,t^{-1}]}$

- On the proof of Abouzaid's Thm
  Relies on an algebraic lemma:
  Lemma: Let $\mathcal{C}$ be an A_∞-category (assume $\mathcal{C} \underset{A_\infty}{\simeq} D^\pi \mathcal{C}$). And
  fix $\mathcal{B} \subset \mathcal{C}$ a full subcategory. (Note $D^\pi \mathcal{B} \subseteq D^\pi \mathcal{C} = \mathcal{C}$).
  Then $K \in \mathrm{ob}\,\mathcal{C}$ is in $D^\pi\mathcal{B}$ if the map
  $$\mathcal{Y}_K^R \underset{\mathcal{B}}{\otimes} \mathcal{Y}_K^L \longrightarrow \mathrm{End}(K)$$
  hits the unit $[1] \in H^0\mathrm{End}(K)$.

  What is this?
  Defn: A left module over an A_∞-category $\mathcal{B}$ is an A_∞ functor
  $$\mathcal{B} \longrightarrow \mathrm{Chain}_{\mathbb{Z}} \qquad \text{(cf exercise)}$$
  A right module over $\mathcal{B}$ is an A_∞ functor
  $$\mathcal{B}^{op} \longrightarrow \mathrm{Chain}_{\mathbb{Z}}$$

  Can now $\otimes$ two modules over $\mathcal{B}$
  Defn: If L and R are left and right modules over $\mathcal{B}$.
  Then $R \otimes_{\mathcal{B}} L \in \mathrm{Chain}_{\mathbb{Z}}$
  has underlying abelian group... (next page)

(defn, continued)

$$\bigoplus_{X_0,\dots,X_n \in \mathrm{Ob}\,\mathcal{B}} R(X_0) \otimes_{\mathbb{Z}} \hom_{\mathcal{B}}(X_0, X_1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \hom_{\mathcal{B}}(X_{n-1}, X_n) \otimes L(X_n)$$

where an element

$$r_0 \otimes x_{01} \otimes \cdots \otimes x_{n-1,n} \otimes \ell_n$$

has degree

homologically: $|r_0| + |\ell_n| + \sum (\deg(x_{i,i+1}) + 1)$

co — : $|r_0| + |\ell_n| + \sum (\deg(x_{i,i+1}) - 1)$

Digression: Compare to the following..

Fix algebra A, left mod L, right mod R

Then how to compute $\mathrm{Tor}_\bullet^{A/k}(L, R)$?

Take the bar complex

$$R \otimes_k L \leftarrow R \otimes_k A \otimes_k L \leftarrow R \otimes_k A \otimes_k A \otimes_k L \leftarrow \cdots$$

with differential

$$\begin{aligned} r \otimes a_1 \otimes \cdots \otimes a_n \otimes \ell \longmapsto\ & r \cdot a_1 \otimes a_2 \otimes \cdots \otimes a_n \otimes \ell \\ & - r \otimes a_1 a_2 \otimes a_3 \otimes \cdots \otimes a_n \otimes \\ & + r \otimes a_1 \otimes a_2 a_3 \otimes a_4 \otimes \cdots \otimes a_n \otimes \ell \\ & - \cdots \\ & + (-1)^n r \otimes a_1 \otimes \cdots \otimes a_n \cdot \ell \end{aligned}$$

This has $\partial^2 = 0$ and homology

$$H_\bullet(B_A(L,R)) \cong \mathrm{Tor}_\bullet^{A/k}(L, R)$$

So now, differential on $R \otimes_{\mathcal{B}} L$ is clear.

(Induced by actions, not going to write formula, but involves
- μ^k terms for $\mathcal{B}$
- the data making L, R modules)

So finally..

Defn: $\mathcal{Y}_K^L$ is the functor $\mathcal{B} \longrightarrow \mathrm{Chain}_{\mathbb{Z}}$ given by (co) Yoneda

$$X \longmapsto \hom_{\mathcal{B}}(K, X) \simeq \hom_{\mathcal{C}}(K, X)$$ (corepresented by K)

$$\mathcal{Y}_K^R : \mathcal{B}^{op} \longrightarrow \mathrm{Chain}_{\mathbb{Z}}$$ (represented by K)

$$X \longmapsto \hom_{\mathcal{B}}(X, K)$$

In the special case $L = \mathcal{Y}^L_K$, $R = \mathcal{Y}^R_K$, get map

$$R \otimes_{\mathcal{B}} L \longrightarrow \mathrm{End}(K)$$

for any full subcategory $\mathcal{B} \subseteq \mathcal{C}$ $(\ni K)$

What is this map?

$$\hom(K, X_0) \otimes \hom(X_0, X_1) \otimes \cdots \otimes \hom(X_{n-1}, X_n) \otimes \hom(X_n, K)$$

$$\Big\downarrow \text{compose} \quad \left(\text{using a lot of } \mu^k \text{ terms}\right)$$

$$\mathrm{End}(K) \; (= \hom(K, K))$$