1 October 9.

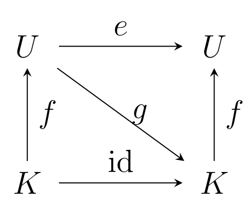
Fixes from last time due to confusion over left vs. right modules:

Lemma 1.1. *Let $\mathscr{B} \hookrightarrow \mathscr{C}$ be a full A_∞ subcategory. Fix $K \in ob(\mathscr{C})$. If $Y^r_K \otimes_{\mathscr{B}} Y^l_K \to Hom_{\mathscr{C}}(K,K)$ hits a unit of $Hom_{\mathscr{C}}(K,K)$, then $K \in D^\pi(\mathscr{B})$. Here $Y^r_K : \mathscr{B}^{op} \to Chain_{\mathbb{Z}}$ is $\cdot \mapsto Hom_{\mathscr{B}}(\cdot, K)$ and $Y^l_K : \mathscr{B} \to Chain_{\mathbb{Z}}$ is $\cdot \mapsto Hom_{\mathscr{B}}(K, \cdot)$, and*

$$Y^r_K \otimes_{\mathscr{B}} Y^l_K = \oplus_{X_0,\dots,X_n \in ob(\mathscr{B})} Y^r_K(X_n) \otimes Hom_B(X_{n-1}, X_n) \otimes \cdots \otimes Hom(X_0, X_1) \otimes Y^l_K(X_0) \in Chain_{\mathbb{Z}}.$$

The map to $Hom_{\mathscr{C}}(K,K)$ is what you would expect given the definitions of Y^r_K, Y^l_K.

Proof. Pass to $Fun(\mathscr{C}^{op}, Chain_{\mathbb{Z}})$. To prove $K \in D^\pi \mathscr{B} \subset D^\pi \mathscr{C} \subset Fun(\mathscr{C}^{op}, Chain_{\mathbb{Z}})$, we need to exhibit an object $U \in D^b \mathscr{B}$ and morphisms $\underline{K} \to U \to \underline{K}$ commuting with $\mathrm{id} : \underline{K} \to \underline{K}$. Here we think of $U, \underline{K}$ as objects in $Fun(\mathscr{C}^{op}, Chain)$, where $\underline{K} : \mathscr{C}^{op} \to Chain_{\mathbb{Z}}$ is $\cdot \mapsto \mathrm{Hom}_{\mathscr{C}}(\cdot, K)$. If we have this, then we have a commutative diagram



where e is an idempotent and (K, f, g) split e. So if $U \in D^b \mathscr{B}$, which is generated from $\mathscr{B}$ by a finite number of direct sums and cones, then $K \in D^\pi B$, which also has idempotents.

How to exhibit such an object? By assumption, $Y^r \otimes_{\mathscr{B}} Y^l \to \mathrm{Hom}(K,K)$ hits a unit. Let $\tilde{u} \in Y^r \otimes_{\mathscr{B}} Y^l$ be an element that hits a unit. Then

$$\tilde{u} \in \oplus_{n \le N} \oplus_{X_0,\dots,X_n} \mathrm{Hom}(X_n, K) \otimes \mathrm{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \mathrm{Hom}(X_0, X_1) \otimes \mathrm{Hom}(K, X_0)$$

for some $N < \infty$. Define

$$U : \mathscr{C}^{op} \to Chain_{\mathbb{Z}}$$

$$\cdot \mapsto \oplus_{n \le N} \oplus_{X_0,\dots,X_n} \mathrm{Hom}(X_n, K) \otimes \mathrm{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \mathrm{Hom}(X_0, X_1) \otimes \mathrm{Hom}(\cdot, X_0).$$

We need to exhibit

$$f \otimes g \in \mathrm{Hom}_{\mathscr{C}^{op}-Mod}(\underline{K}, U) \otimes \mathrm{Hom}_{\mathscr{C}^{op}-Mod}(U, \underline{K})$$

hitting $\mathrm{id}_{\underline{K}} \in \mathrm{Hom}_{\mathscr{C}^{op}-Mod}(\underline{K}, \underline{K})$. By the Yoneda lemma,

$$\mathrm{Hom}_{\mathscr{C}^{op}-Mod}(\underline{K}, U) \cong U(K)$$

$$= \oplus_{n \le N} \oplus \mathrm{Hom}(X_n, K) \otimes \mathrm{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \mathrm{Hom}(X_0, X_1) \otimes \mathrm{Hom}(K, X_0)$$

which contains $\tilde{u}$. Let $f = \tilde{u}$. But there is also an obvious map $U \to \underline{K}$ given by composition: for all $X \in ob(\mathscr{C})$, we need a map

$$\oplus_{n \le N} \oplus \mathrm{Hom}(X_n, K) \otimes \mathrm{Hom}(X_{n-1}, X_n) \otimes \cdots \otimes \mathrm{Hom}(X_0, X_1) \otimes \mathrm{Hom}(X, X_0)$$

$$= U(X) \to \underline{K}(X) = \mathrm{Hom}_{\mathscr{C}}(X, K)$$

which we can take to be $\sum \mu^n$. By assumption, $g \circ f = g \circ \tilde{u}$ is a unit in K.

We're almost done. Problem: is U really in D^b? Tensoring with an infinite rank thing produces an infinite direct sum, which D^b doesn't cover. So if $\mathscr{C}$ has hom complexes with ∞-rank cohomology, U might not be in $D^b\mathscr{B}$.

(Interlude: we have

$$\mathscr{B} \hookrightarrow D^b\mathscr{B}$$

$$L \mapsto \mathrm{Hom}_{\mathscr{C}}(\cdot, L) = \underline{L}.$$

Given a $\mathbb{Z}$-module V, we can create a new element of $\mathscr{C}^{op} - Mod$ by

$$V \otimes \underline{L} : \cdot \mapsto V \otimes \mathrm{Hom}_{\mathscr{C}}(\cdot, L)$$

for $\cdot \in \mathscr{C}$. If $V = \oplus_n A_n$, then $V \otimes \underline{L} = \oplus A_n \otimes \underline{L}$.)

Fortunately, we can arrange for U to be in $D^b\mathscr{B}$ as follows: by definition, $\tilde{u} \in U(K)$ is expressed as a finite sum of elements in $Y^r \otimes Y^l$, so we don't need all of $\mathrm{Hom}(X_i, X_j)$—we can pull out the finitely generated parts of $\mathrm{Hom}(X_i, X_j)$ that contribute to $\tilde{u}$. □

Remark 1. What we've actually shown is not a commutative diagram [$\underline{K} \to U \to \underline{K}$ commuting with $\mathrm{id} : \underline{K} \to \underline{K}$] in $\mathscr{C}^{op} - Mod$, but rather a diagram [$\underline{K} \to U \to K$ commuting with $\mathrm{id} : \underline{K} \to K$] in $H^\bullet\mathscr{C}^{op} - Mod$, which has the same objects as $\mathscr{C}^{op} - Mod = Fun_{A_\infty}(\mathscr{C}^{op}, Chain_{\mathbb{Z}})$ but has morphisms

$$\mathrm{Hom}_{H^\bullet}(F, G) := \oplus_n H^n \mathrm{Hom}_{\mathscr{C}^{op}-Mod}(F, G).$$

So we really exhibited $\underline{K}$ as in some idempotent completion of the $H^\bullet$ category. But a special property of idempotents is that they always lift to honest A_∞ idempotents in the A_∞ category (not the $H^\bullet$ category). This is a special kind of colimit that commutes with passing to the $H^\bullet$ category.

How does Abouzaid use this lemma? Fix some object $B \in PWFuk$ that we suspect to be a generator, or more generally, $\mathscr{B} \subset PWFuk = \mathscr{C}$. There's a geometric criterion for when

$$Y^r_K \otimes_{\mathscr{B}} Y^l_K \to \mathrm{Hom}(K, K)$$

hits the unit of K for any $K \in ob(\mathscr{C})$. This goes through the "symplectic (co)homology", which is an invariant of the symplectic manifold M. The main geometric result is that the following diagram commutes (write $HH(B)$ for the Hochschild homology).

$$\begin{array}{ccc} HH(B) & \longrightarrow & Y^r_K \otimes_{\mathscr{B}} Y^l_K \\ \downarrow & & \downarrow \\ SH(M) & \longrightarrow & \mathrm{End}(K) \end{array}$$

So if $HH(B) \to SH(M) \to \mathrm{End}(K)$ hits the unit, we know B split-generates a category containing K.

Claim: when Q is spin, and $L = T_q^\vee Q \subset T^\vee Q$, then for all $K \in PWFuk(T^\vee Q)$, $Y_K^r \otimes_L Y_K^l \to \mathrm{End}(K)$ hits the unit. Consequently, when Q is spin,

$$D^\pi WFuk(T^\vee Q) \cong \mathrm{End}(T_q^\vee Q) - Mod \cong D^\pi(T_q^\vee Q)$$

where in the last term, $T_q^\vee Q$ is acted on by $\mathrm{End}(T_q^\vee Q)$.

Theorem 1.2 (Abouzaid)**.** *When Q is spin,* $\mathrm{End}^\bullet(T_q^\vee Q) \cong C_\bullet \Omega Q$*, where ΩQ is the space of loops based at q, that is,*

$$\{\gamma : [0,1] \to Q \mid \gamma(0) = \gamma(1) = q\}.$$

Remark 2. In this case, the symplectic topology of $T^\vee Q$ reduces totally to algebraic topology.

Note that we have

$$\Omega Q \times \Omega Q \to \Omega Q$$

$$(\gamma, \tilde{\gamma}) \mapsto \tilde{\gamma} \cup_q \gamma.$$

Also, one can convert a homology chain to a cohomology chain by $A_\bullet \mapsto A^{-\bullet}$.

This tension between cohomology and homology objects is not so rare. For example, let R be a smooth commutative ring over a perfect field k.

Theorem 1.3 (Hochschild-Kostant-Rosenberg)**.**

$$HH_\bullet(R) \cong \Omega^\bullet_{R/k}.$$

Exercise 1.1. Check what we've said with our example $Q = S^1$.