Kevin Sackel — Fukaya Categories - L16 — 14 Oct 2015

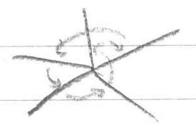
Hiro's not here -- Fabian Haiden on Ribbon Graphs

- Note:

$G \subset S^2$ genus 0 $\qquad$ $G \subset T^2$ genus 1

How to distinguish?

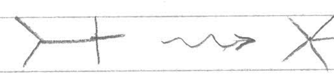
Look at cyclic ordering of half edges

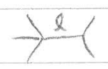


(Ternary relation: given 3, are they clockwise are counterclockwise)

- Defn: Ribbon graph is this data...

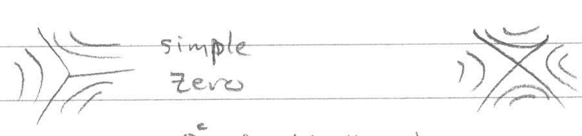
- Note: Ribbon graph $\rightsquigarrow$ surface w/ boundary
  genus, # disks removed

- Defn: Equivalence of g, n: isotopy + 

- Note: Have metric  $l \in \mathbb{R}_{>0}$ $\rightsquigarrow$ metrized ribbon graph

- Thm [Strebel]
  $$\underbrace{\mathbb{R}^n_{>0}}_{\text{lengths of bdry}} \times \mathcal{M}_{g,n} = \text{moduli space of such graphs corresponding to surface of genus } g \text{ and } n \text{ bdry components}$$

- Note on Thm: S = compact Riemann surface
  Meromorphic quadratic differentials $\phi \in \Gamma(S, K^2)$ $\quad$ ($f(z)dz^2$ locally)
  $\rightsquigarrow$ horizontal foliation $\phi(v,v) \in \mathbb{R}_{>0}$



simple zero $\qquad$ double zero

Strebel: $\exists!$ mero^c quad. diff. w/ double poles s.t. almost all trajectories closed specifying residues $\in \mathbb{R}^n_{>0}$

$\bigcup$ non-closed trajectories + zeroes $= \Gamma \rightsquigarrow$ metrized ribbon graph

§ Fukaya Categories of Surfaces (Idea: Kontsevich)

- Note: $\Gamma \subset S$, S surface w/ punctures
  Γ is retract of S, ribbon graph
  (Kontsevich proposal: In higher dims, Γ = Lagrangian skeleton)

- Claim: $\exists$ (co)sheaf of A_∞-categories on Γ s.t. the global sections form the Fukaya category of S.

- Note:
  Sheaf - locally constant on edges (constructible)

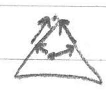
  Concretely:

  Vertex $v \rightsquigarrow \mathcal{C}_v$ = A_∞-category

  Edge $e \rightsquigarrow \mathcal{C}_e$ = A_∞-category

  Restriction maps

   $\mathcal{C}_v \to \mathcal{C}_e$ for each half-edge

  Rmk: No conditions here, but higher dims, need coherence $\rightsquigarrow$ 

  Cosheaf: $\mathcal{C}_e \to \mathcal{C}_v$ instead.

- Note: Duality for A_∞-categories

  $$\mathcal{A} \longmapsto \mathcal{A}^* = \mathrm{Fun}(\mathcal{A}, \underbrace{\mathrm{Perf}(\mathbb{Z})}_{\text{complexes of AbGps with finite rk cohomology}})$$

  Sheaf $\longmapsto$ Cosheaf

  (not involution in general)

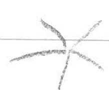
  Have $\mathcal{A} \cong \mathcal{A}^* \iff \mathcal{A}$ is smooth + proper in homological sense

- Note: Cosheaf on ribbon graph:

  $\mathcal{C}_e = \mathrm{Perf}(\mathbb{Z}) = \mathrm{Fuk}(pt) \overset{\text{Morita}}{\cong}$ Category w/ 1 object and End $\cong \mathbb{Z}$

  $\rightsquigarrow$ Perf(A) = Perf(B) as A_∞-categories

  $\mathcal{C}_v$ = Fuk (disk w/ val(v) marked pts)

  valency n

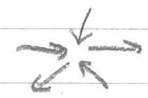
  $\rightsquigarrow$ A_∞-category

  Objects = X_i, $i \in \mathbb{Z}/n\mathbb{Z}$ $\rightsquigarrow$ corresponds to ribbon graph data

  Morphisms = $1_{X_i} \in \mathrm{End}(X_i)$, $a_i \in \mathrm{Hom}(X_i, X_{i+1})$

More on $\mathcal{C}_v$... assume $\mathbb{Z}/2\mathbb{Z}$ grading for simplicity

Choose orientation of each edge



$x_i \searrow \nearrow x_{i+1} \Longrightarrow \deg(a_i) = 1$

$x_i \nwarrow \nearrow x_{i+1} \Longrightarrow \deg(a_i) = 0$

A_∞-struct? $\mu^2(1_{x_i}, a_{i-1}) = a_{i-1}$

$\mu^2(a_i, 1_{x_{i-1}}) = a_i$

$\mu^n(a_{i-n+1}, \ldots, a_i) = 1_{x_i}$

and all other $\mu^* = 0$

Finally, corestrictions...

Need $\mathcal{C}_e \cong \dfrac{\{*\}}{\mathrm{End}\,* = \mathbb{Z}} \longrightarrow \mathcal{C}_v$

$\{*\} \longmapsto X_i$ with i corresponding to half edge

Gives a cosheaf!

Dual sheaf:

$$\begin{array}{ccc} \mathcal{C}_v^* & \longrightarrow & \mathcal{C}_e^* \\ \| & & \| \\ \mathrm{Fun}(\mathcal{C}_v, \mathrm{Perf}(\mathbb{Z})) & & \mathrm{Fun}(\mathcal{C}_e, \mathrm{Perf}(\mathbb{Z})) \\ \simeq & & \\ \mathcal{C}_v & & \end{array}$$

Two categories

computing global sections is htpy colimit

$\Gamma(\text{sheaf}) \subset \Gamma(\text{cosheaf})$

Fuk with cpt Lagr — Wrapped Fuk

this is the better one.