302

10/10 Math-277 (TFT and Hochschild)

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exercise let

$$Z: Cob^{\amalg} \to Vect_k^{\otimes}$$

be a TFT. Show that $Z(pt)$

must be a finite dim vector space.

Today: intro to TFTs.

Defn. An n-dim topo. field theory

is a symmetric monoidal

functor $Z: ((Cob_n^{fr})^{\amalg} \to \mathcal{C}^{\otimes}$ ("partition function")

where $\mathcal{C}^{\otimes}$ is some (∞, n)

- category w/ $\otimes$ ⇝ symmetric

monoidal lecture.

Rmk. There's a thm, now proven (Ayala - Francis) that gives an algebraic characterization of all possible Z.

Thm ("cobordism hypothesis" - á la Baez-Dolan, Hopkins-Lurie ...)

$\exists$ an equiv of topo. spaces

$$\left\{ Z \right\} \simeq \left\{ \begin{array}{l} \text{moduli of fully} \\ \text{dualizable objects of } \mathcal{C} \end{array} \right\}$$

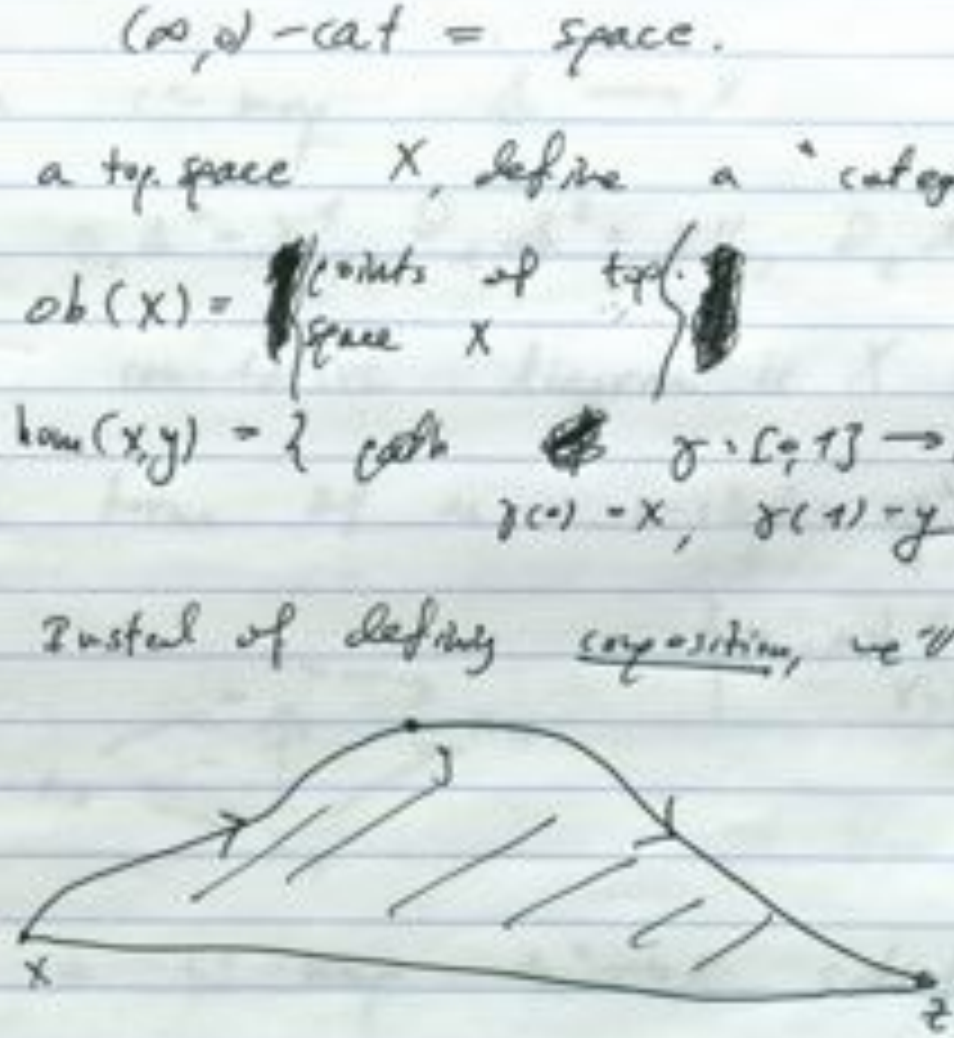
The proof is completely different from Lurie strategy. Chris is trying to follow the original program.

<u>(∞,n)-cat.</u> For your sanity, all definitions are heuristic.

<u>$n=0$.</u> $(\infty,0)$-cat = space.

Given a top. space X, define a "category" X:

- $ob(X) = \{$points of top. space $X\}$
- $hom(x,y) = \{$path $\gamma : [0,1] \to X$ | $\gamma(0) = x$, $\gamma(1) = y\}$
- Instead of defining <u>composition</u>, we'll



say what a <u>commutative</u> <u>diagram</u> should be!

A commutative triangle

$$\begin{tikzcd} & y \arrow[dr, "\gamma'"] & \\ x \arrow[ur, "\gamma"] \arrow[rr, "\gamma''"'] & & z \end{tikzcd}$$

is a C^0-map $\Delta^2 \to X$

s.t. $\partial_0 \Delta^2 = \gamma'$, $\partial_1 \Delta^2 = \gamma''$, $\partial_2 \Delta^2 = \gamma$.

. A commutative diagram is X in

the form of an n-simplex $\Delta^n \subset \mathbb{R}^{n+1}$

$$\left\{ (x_0, \dots, x_n) \,\middle|\, \sum x_i = 1, \ \forall i, \ 0 \le x_i \le 1 \right\}$$

$$\begin{tikzcd} & x_2 \arrow[r] & x_4 \\ x_0 \arrow[ur] \arrow[r] \arrow[urr, dashed] & x_1 \arrow[u] \arrow[ur] & \end{tikzcd}$$

is a C^0-map $\Delta^n \to X$ satisfying

the obvious boundary conditions.

(The obvious inclusions $\mathbb{R}^n \to \mathbb{R}^{n+1}$ induce

$\Delta^{n-1} \to \Delta^n$ and $\partial^j \Delta^n \subset \Delta^n$ i.e. delete j^{th} vertex + interior.

$\{ (x_0, \dots, x_n) \in \Delta^n / x_j = 0 \}$

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$\triangle$ Fixing two morphisms

$$\gamma : x \to y$$

$$\gamma' : y \to z$$

there exist ∞-many commutative triangles

$$\begin{tikzcd} & y \arrow[dr, "\gamma'"] & \\ x \arrow[ur, "\gamma"] \arrow[rr, "\gamma''"'] & & z \end{tikzcd}$$

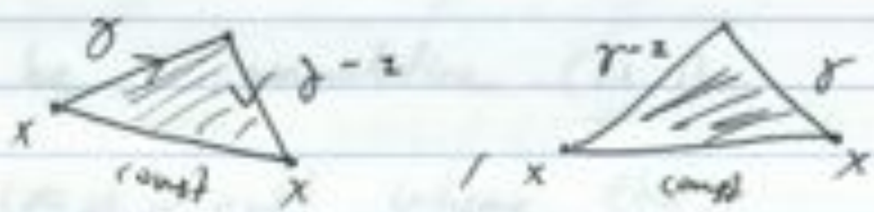
However, we can prove that the space of such triangles is <u>contractible</u>. (follows from weak Kan condition)

<u>Rmk</u> every morphism in X is invertible up to homotopy.

In <u>classical</u> category theory (e.g. Maclane)

$f : x \to y$ is invertible $\Longleftrightarrow$ $\begin{tikzcd} & y \arrow[dr, "f^{-1}"] & \\ x \arrow[ur, "f"] \arrow[rr, "id_x"'] & & x \end{tikzcd}$, $\begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "f^{-1}"] \arrow[rr, "id_y"'] & & y \end{tikzcd}$

exist ($\exists f^{-1}$ and these are commutative triangles). This in our case is obvious:

$\exists$ 

For that reason, an $(\infty,0)$-cat is an ∞-groupoid. The "0" in $(\infty,0)$-category is the fact that every k-morphism for $k>0$ is invertible.

__n=1.__ An $(\infty,1)$-category is

- A category where
- $\hom(x,y)$ is a topo. space (i.e. a category enriched in top. spaces)

or

- the data of what it means for a k-simplex to be commutative (as in $(\infty,0)$-cat) where the edges NEED NOT be invertible.

comment. space of possible compositions is still contractible.

Example. Let X be a space w/ stratification: (Jacob's defn). a space X with a continuous map $X \to P$ where P is a poset. P is topo so that $U \subseteq P$ open $\iff$ U is upward closed i.e. if $x \in \mathcal{U}$, then any $y \geq x$ is in $\mathcal{U}$.

<u>Sub Example.</u> $X = \phi$, $P = \phi$

Given $X \to P$ stratified, consider

$$C(X) := \begin{tikzcd} X \arrow[r, hook] \arrow[d] & X \times [0,\infty) \arrow[d] \arrow[dl, phantom, "\lrcorner", very near start] \\ * \arrow[r] & C(X) \end{tikzcd}$$

the <u>open cone</u>. When $X \neq \emptyset$ this is

$$X \times [0,\infty) \big/ X \times \{0\}$$

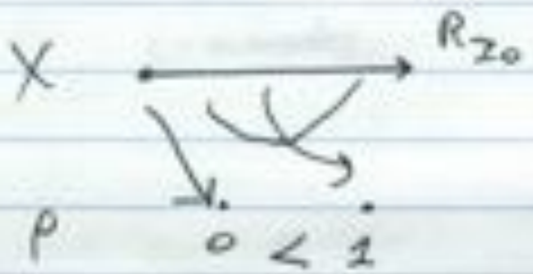
and $C(P) = P$ w/ new minimal element adjoined.

Sub-Example. (1) If $X = P = \emptyset$, then $C(X) = pt$

and $C(P) = pt$, so

$$\begin{array}{ccc} C(X) & = & pt \\ \downarrow & & \downarrow \\ C(pt) & = & pt \end{array}$$

(2) $X = P = pt$



space X is now two pieces.

(3) Repeat ...,

(3)

$$\begin{array}{c} \Delta^n \\ \downarrow \\ 0<1<\cdots<n \end{array}$$

The example (finally!) is the Ex. 7

path category of a stratified space X.

objects = points of X

$\mathrm{mor}(x,y)$ = space of continuous functions $C(*) = \mathbb{R}_{\geq 0} \xrightarrow{\gamma} X$ $\quad \gamma(0)=x$, $\gamma(1)=y$

s.t. diagram

$$\begin{tikzcd}
C(*) = \mathbb{R}_{\geq 0} \arrow[r, "\gamma"] \arrow[d] & X \arrow[d] \\
0 < 1 \arrow[r] & P
\end{tikzcd}$$

map of poset (= continuous map of posets)

commutes.

A comm. diagram in shape of Δ^k is a C^0-map

$$\begin{tikzcd}
\Delta^k \arrow[r] \arrow[d] & X \arrow[d] \\
0 < \dots < k \arrow[r] & P
\end{tikzcd}$$

Example. $X = \overset{x_0 \quad 1 \quad y_0}{\underset{0 \quad z \quad 0}{\bullet\!\!-\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\bullet}}$

$$\downarrow$$

$$P = \{0<1\}$$

Claim. $\hom(x_0, y_0) = \phi$. Consider

$$\begin{tikzcd} R_{z_0} \arrow[r, "\gamma"] \arrow[d] & X \arrow[d] \\ 0<1 \arrow[r] & 0<1 \end{tikzcd}$$

Also, a path from x_0 to z is not invertible.

Example. Cob_1^{fr}: Objects are disjoint union of mflds of the form $pt \times R$ + orientations.

You should ignore the R-component. Think its + infinitesimal nbhd + orientations.

A morphism is an oriented cobordism.

<u>Example.</u> $X = \begin{matrix} \longrightarrow + \\ \longleftarrow - \end{matrix}$, $Y = \begin{matrix} \longrightarrow + \\ \longleftarrow - \end{matrix}$

Some morphism X to Y:

id:

$X \times \mathbb{R}$

Composition is as usual.

Note Cob_1^{fr} has a symmetric monoidal structure given by disjoint union.

<u>$n \geq 2$.</u> An (∞, n)-category is a gadget with

- objects $x, y, \ldots$

(1-) • morphisms $\underset{f,\,g}{\hom(x,y)}$, ...

(2-) • morphisms <u>between</u> morphisms $\hom(f,g)$

$\vdots$

. till we have a space of n-morphism.

<u>Example.</u> $\mathcal{C}$ = cat of categories

ob = X is a category

$\hom(X,Y) = \{ \text{functor } F : X \to Y \}$

$$2\hom(F,G) = \left\{ \begin{tikzcd} X \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \end{tikzcd} \right\} \quad \left\{ \begin{array}{l} \text{the set} \\ \text{of nat.} \\ \text{trans.} \\ \eta : F \Rightarrow G \end{array} \right\}$$

<u>Example.</u> $\mathcal{C}$ = Morita category

ob = X is an assoc algebra

$\hom(X,Y) = \left\{ \begin{array}{c} (X,Y)\text{-bimodules} \\ M \end{array} \right\}$

$2\hom(M,N) = \left\{ \begin{array}{c} \text{bimodule} \\ \text{maps} \end{array} \right\}$

$$\begin{array}{ccc} & M & \\ X & & Y \\ & N & \end{array}$$

There is a functor from one to the other called "take modules".