Kevin Sackel — Fukaya Categories, Harvard Math 277 — L18 — 16 Oct 2015

Hiro not here – Siu-Cheong Lau

on Introduction to SYZ Mirror Symmetry for Toric CY Mflds

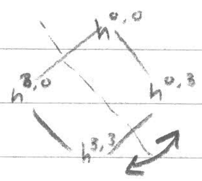
• Defn: Calabi-Yau: $(\underbrace{X^{2n}, \overbrace{J, \omega, g}^{\text{compatible}}}_{\text{Kähler}}, \Omega)$ ← nowhere vanishing holo^c n-form

s.t. $\omega^n = \Omega \wedge \bar{\Omega}$ (Calabi-Yau condition)

• Note: Historically, mirror symmetry was about mirror pair of CY mflds $X, \check{X}$

First level: Hodge diamonds

X $\check{X}$



flipping gives Hodge diamond for $\check{X}$

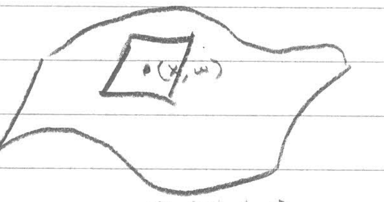
(Physicists discovered always for CY 3-dim$_{\mathbb{C}}$ mflds)

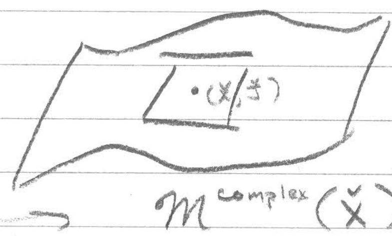
Note: $h^{2,1}$ = deformation of complex structure

So $h^{2,1}(\check{X}) \stackrel{\circ}{=} \underbrace{H^1(T_{\check{X}})}_{\text{[Kodaira-Spencer]}} \underset{\text{contract}}{\simeq} H^1(\Omega^{n-1}) = H^{n-1,1}$

But $h^{2,1}(\check{X}) = h^{1,1}(X)$ ... $[\omega] \in H^{1,1}$

↪ deformations of Kähler structure





$\mathcal{M}^{\text{Käh}}(X) \longleftrightarrow \mathcal{M}^{\text{complex}}(\check{X})$

tangent spaces identified

↪ as Frobenius structures

Closed-string mirror symmetry

① "Flat coords" on each side

A-period $\qquad$ identify $\qquad$ B-period

$\omega \longmapsto \int_{[C]} \omega \quad \longleftrightarrow \quad \Omega \longmapsto \int_{[L]} \Omega$

$[C]$: holo^c curves $\in H_2^{\text{effective}}$ $\qquad$ $[L]$: Lagrangian $\in H_n$

Claim: ∃ identification basis of holo^c curves & Lagrangian cycles

More structure:

(2) Yukawa couplings (on tangent space)

A-side: $(u * v, w)$ — $*$: quantum cup product; $(,)$: Poincaré pairing

B-side: $\int_{\check{X}} (\partial_u \partial_v \partial_w \Omega^{3,0}) \wedge \Omega^{3,0}$

These are equal under association of tangent space.

• Note

(3) Homological mirror symmetry:

identifies $C_i \longleftrightarrow L_i$

In fact:

identifies: coherent sheaves $\longleftrightarrow$ Lagrangian submflds

• Note:

deeper $\longrightarrow$

Hodge diamond | closed-string MS ($g=0$) | HMS

higher genus mirror symmetry $\hookrightarrow$ quantize the moduli space (not very well understood)

• Example (not CY)

Kähler $(\mathbb{P}^1, \omega_{FS}, J) \rightsquigarrow (\mathbb{C}^*, W = z + \frac{q}{z})$ — W: potential

$H^* = \text{span}\{\mathbb{1}, pt\}$ $\quad Jac(W) = \mathbb{C}[z, z^{-1}] / \partial_z W \longrightarrow \boxed{z^2 = q}$

Quantum Product: $\mathbb{1}$ = unit

$\boxed{pt * pt = q} = e^{-\int_{\mathbb{P}^1} \omega}$

$\boxed{\begin{array}{l} \mathbb{1} \longmapsto 1 \\ pt \longmapsto z \end{array}}$ $\hookrightarrow$ gives IM identifying products

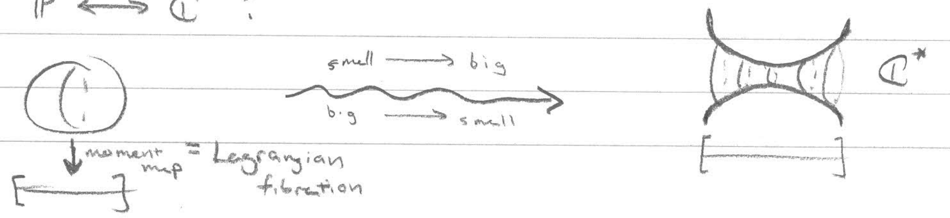
Q: How does this mirror $\mathbb{P}^1 \longleftrightarrow \mathbb{C}^*$ arise?

• Note: Strominger - Yau - Zaslow (SYZ)

Mirror symmetry is T-duality up to first order

T-duality = torus-duality (linear algebra): $T = V/\Lambda \longleftrightarrow T^* = V^*/\Lambda^*$

• Example: $\mathbb{P}^1 \longleftrightarrow \mathbb{C}^*$?



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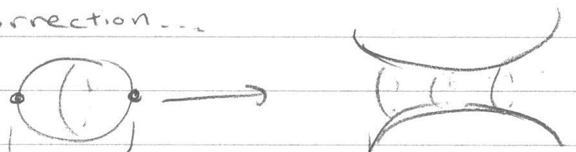
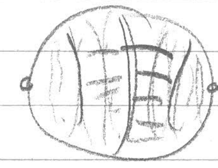
- Remark: Correction...



info lost ⇝ how to correct? Potential!

$$\text{"} W = \sum_\beta n_\beta q^\beta z^\beta \text{"}$$

↳ holo^c disc classes emanating from singular fibres

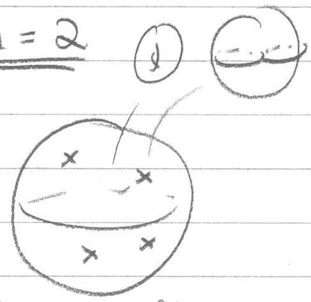


$= z + q/z$ in this case

(torus) ⇝ automatic by Arnol'd

- Note: For SYZ, need Lagr. fibration over CY, e.g. S^n

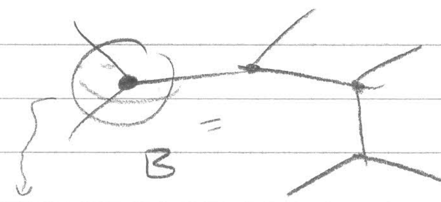
E.g. $n = 2$



24 singular fibers (discriminant locus)

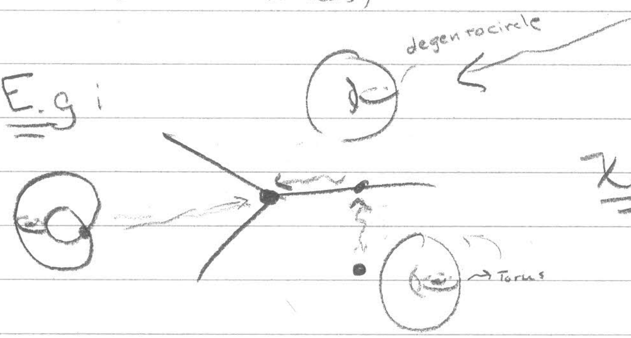
$n=3$

codim (discriminant locus) = 2

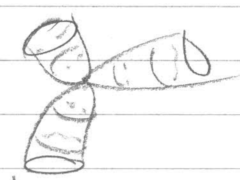


why things are hard "Y-vertex" ⇝ different kinds

E.g.:

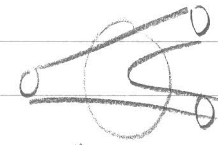


$\chi = 1$



topological mirror

$\chi = -1$



non-trivial

(Ruan - Gross)

"tropical geometry" — amoeba —