10/21.

Exer. Show that a monoidal (not nec. symmetric) category is the same thing as a 2-category with one object.
(Compare: A monoid is a category w/ one object)

Defn. A n-dimensional (framed) topological field theory is a symm mon functor

$$Z: (Cob_1^{fr})^{\amalg} \longrightarrow \mathcal{C}^{\otimes}$$

where $\mathcal{C}^{\otimes}$ is a symm mon. $(\infty, n)$-category.

Last time: An $(\infty, n)$ category is a topological space.

"Defn": An $(\infty, n)$-category is a category enriched in a $(\infty, n-1)$-categories.

Ex. $(n=2)$ the "category" of $(\infty,1)$-cats is a $(\infty, 2)$-cat.

Obj $\ni \mathcal{C}$ an $(\infty,1)$-cat.

$$\hom(\mathcal{C}, \mathcal{D}) := \underset{\text{an } (\infty,1)\text{-category}}{\underset{\uparrow}{\mathrm{Fun}}}(\mathcal{C}, \mathcal{D})$$

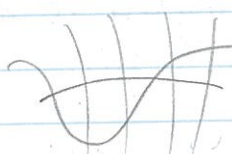
Ex: Morita cats

Obj: associative $(A_\infty)$-algebras

$\hom(A, B) =$ cat of $A$-$B$ bimod ($\infty$-cat).

Rmk. One reason for this vague discussion is to remain "model-independent". If you have a defn of $(\infty, n)$-cat and so does mathematician X, and if both defns are morally correct, then regardless of the defn (i.e., the model), this discussion is valid.

Ex. (of the philosophy)
What <u>is</u> the moduli space of holom. strips between $L_0, L_1 \subset M$?

(i) 

$\mathrm{Hom}^{0,1}(\ \ )$

$\bar\partial$

$C^\infty(\mathbb{R}\times[0,1], M)$

$\mathrm{graph}(\bar\partial) \cap$ zero section. (<u>not transverse!</u>)

perturb (Floer, Seidel)

$\mathrm{graph}(d+v)^{0,1} \cap$ zero section

derive

??

/Tor

Kuranishi theory

polyfolds

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"Defn of $\mathrm{Cob}_n^{fr}$"

An object of $\mathrm{Cob}_n^{fr}$ is a 0-mfld to a trivialization of its n-stable tangent bundle.

$\emptyset$ $\qquad \emptyset \times \mathbb{R}^n$

$\cdot$ $\qquad \cdot \times \mathbb{R}^n$ $\qquad T(\cdot\ \cdot) \times \mathbb{R}^n) \overset{\Phi}{\cong} \mathbb{R}^n$

$\cdot\ \cdot$ $\qquad (\cdot\ \cdot) \times \mathbb{R}^n$ $\qquad$ trivial bundle

$\cdot\ \cdot\ \cdot$

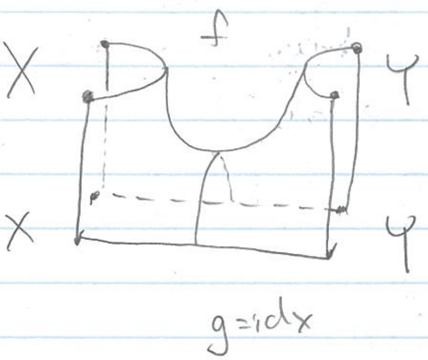
A 1-morphism is a cobordism between 0-mflds w/ an $(n-1)$-stable framing compatible w/ those of the 0-mflds.

$\times \mathbb{R}^{n-1}$

composition : obvious thing

A 2-morphism?

Fixing two objects $X, Y \in Cob_n^{fr}$, a 2-morphism between $f, g$ $(f, g \in \hom(X, Y))$ respecting $X, Y$, w/ stable framing.




saddle.

Rmk. Everything compact.

2-morphism are mflds w/ corners

e.g.


is a cobordism from $\emptyset$ to $\emptyset$, so is a 1-morphism.

$\emptyset_1$ is a cobordism from $\emptyset_0$ to $\emptyset_0$.

So $\emptyset_1$ is a 1-morphism.

$\emptyset_1$  $\emptyset_1$ is a cobordism from $\emptyset_1$ to $\emptyset_1$.

$T^2$

So $T^2$ is a 2-morphism.

A k-morphism between two (k-1)-morphisms f, g is a cobordism from f to g respecting $\partial f$, $\partial g$.

This category: algebraic data of how to glue mflds to give mflds.

Rmk. $Cob_n^{fr}$ has symmetric monoidal structure called $\sqcup$.

Q: What do TFTs look like?

They give invariants of mflds. Also targets

Thm. Let $\mathcal{C}$ be the $(\infty,1)$-category where

$ob\,\mathcal{C}$ = associative alg/k.

$hom(A,B) = \{A\text{-}B \text{ bimodules}\}$.

↑

space

2-simplex
$$\begin{tikzcd}
 & & N' \\
M \arrow[urr, "\cong"] \arrow[drr, "\cong"'] \arrow[rr, phantom, "?"] & & \\
 & & N \arrow[uu, "\cong"']
\end{tikzcd}$$

$\infty$-groupoid of bimodules

Any object A uniquely determines a 1-D TFT s.t. $Z(pt) = A$.

Details: $\mathrm{Cob}_1^{fr} \xrightarrow{Z} \mathcal{C}^{\otimes}$

Since $Z$ is symm mon. it sends $\sqcup \longrightarrow \otimes$

$$\mathrm{unit}_{\sqcup} \longmapsto \mathrm{unit}_{\otimes}$$
$$\mathrm{unit}_{\sqcup} = \emptyset, \qquad \mathrm{unit}_{\otimes} = k \leftarrow \text{ground ring}$$

$$\cdot \sqcup \cdot \sqcup \cdot \sqcup \cdots \sqcup \cdot \longmapsto A \otimes_k A \otimes_k A \otimes_k \cdots \otimes_k A.$$

 $\longmapsto A$

 $\longmapsto A^{op} = A$ as a group

but $A \otimes_k A \xrightarrow{m} A$

$\Downarrow$

$A^{op} \otimes_k A^{op} \xrightarrow{m^{op}} A$

$a \otimes b \longmapsto (-1)^{|b||a|} m(b \otimes a)$

 $\longmapsto {}_A A_A \leftarrow$ A-A bimodule $A$

b/c ${}_A A_A M_B$

$\Downarrow$ composition

${}_A A \otimes_A M_B = {}_A M_B.$

 $\longmapsto \begin{matrix} A \\ \otimes \\ A^{op} \end{matrix} \quad A \quad k$

$\rightarrow\!\bullet \sqcup \leftarrow\!\bullet \longmapsto \emptyset$

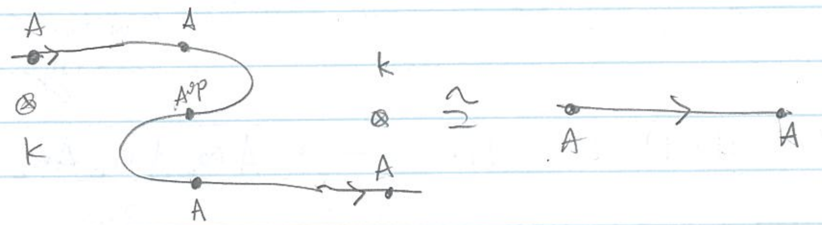
$$(A \otimes A^{op}) \otimes A \longrightarrow A$$
$$(a \otimes b) \otimes x \longmapsto axb.$$

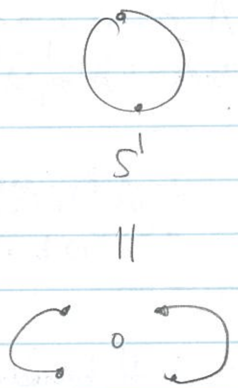
Likewise

 $\xmapsto{Z} {}_k A_{A \otimes A^{op}}$

$\emptyset \longmapsto \rightarrow\!\bullet \sqcup \leftarrow\!\bullet$

Rmk. Uniqueness follows from cleverly applying "Zorro's Lemma"




$$\xmapsto{\ Z\ } \quad Z(S^1)$$

$$\cong$$

$${}_kA_{A\otimes A^{op}} \circ {}_{A\otimes A^{op}}A_k$$

$$\cong$$

$$A \otimes_{A\otimes A^{op}} A$$

Definition. Let $A$ be an $A_\infty$-alg. Its <u>Hochschild homology</u> is the homology of the chain complex

$$A \overset{\mathbb{L}}{\otimes}_{A\otimes A^{op}} A$$

← derived.

i.e. $\mathrm{Tor}^{A\otimes A^{op}}(A,A)$