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Harvard Math 277 (Fukaya Categories) 23 Oct 2015

– L21

- **Exercise.**

  Let X be an ∞-groupoid. Fix $x \in X$. Convince yourself that

  $$\hom(x,x) \simeq \Omega X$$

  homotopy equivalence ↳ based loopspace of X at x.

- **Last Time:** $\mathrm{Cob}_1^{fr} \longrightarrow \mathrm{Morita}$

  $$\mathrm{pt} \longmapsto A = \underset{A_\infty}{\text{assoc.}} \text{ algebra}$$

  $$S^1 : \bigcirc \longmapsto HC_* A = A \overset{\mathbb{L}}{\underset{A \otimes A^{op}}{\otimes}} A$$

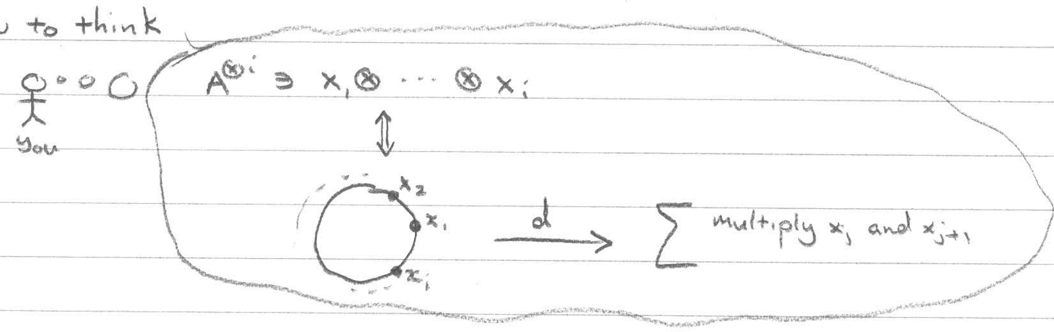
  Hochschild chains of A

  $$HC_* A \simeq A \longleftarrow A \otimes A \longleftarrow A \otimes A \otimes A \longleftarrow \cdots$$

  $$ab - (-1)^{ba} ba \longleftarrow a \otimes b$$

  $$\begin{pmatrix} ab \otimes c - (-1)^{*} a \otimes bc \\ + (-1)^{*} ca \otimes b \end{pmatrix} \longleftarrow a \otimes b \otimes c$$

How to think



- **Note:** Last time, claimed $\exists\, S^1$-action on $HC_*(A)$.

  Reason:

  $$S^1 \in \hom^1_{\mathrm{Cob}}(\emptyset_0, \emptyset_0)$$

  - Since Cob_1^{fr} is an $(\infty,1)$-category, for any two objects X, Y, $\hom(X,Y)$ is an $(\infty,0)$-category, i.e. a space.
  - If $Z : \mathrm{Cob}_1^{fr} \longrightarrow \mathcal{C}$ is a functor, then $Z(X)$ always has an action of $\mathrm{Aut}(X)$
  - So let's compute $\mathrm{Aut}(S^1)$

    Exercise $\rightsquigarrow$ $\mathrm{Aut}(S^1) \simeq \Omega \hom(\emptyset_0, \emptyset_0)$

    ↳ based at S^1

Now, one model for $\hom(\emptyset_0, \emptyset_0)$

is $\{\text{compact 1-dim'l mflds in } \mathbb{R}^\infty\}$

$$= \coprod_{[X]} \mathrm{Emb}(X, \mathbb{R}^\infty)/\mathrm{Diff}(X) \simeq \coprod_{[X]} B\mathrm{Diff}(X)$$

Since $\Omega BG \simeq G$, we have

$$\mathrm{Aut}(S^1) \underset{\text{h.e.}}{\simeq} \mathrm{Diff}(S^1)$$

- Rmk: Say $\mathcal{C}^\otimes = A_\infty\mathrm{Cat}$ ⇝ obj A_∞ categories w/ colims ($\oplus$, cones, unions, idempotents)

  mor functors preserving colims

  Thus,

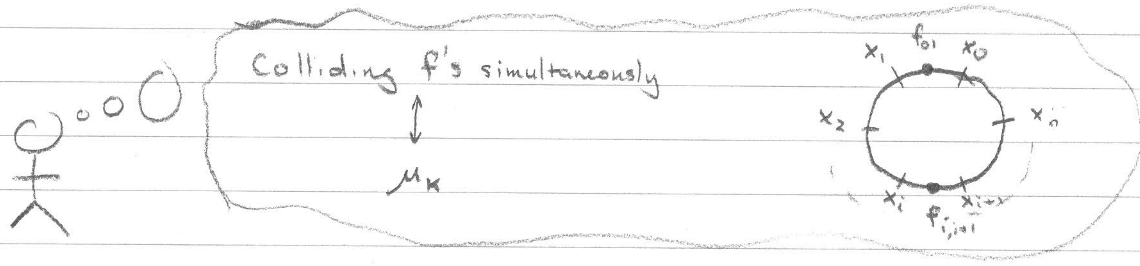
  if we have a functor

  $$\mathrm{Cob}_1^{fr} \longrightarrow A_\infty\mathrm{Cat}^\otimes$$
  $$pt \longmapsto \mathcal{A}$$

  then S^1 is sent to the Hochschild chain complex of $\mathcal{A}$.

- Defn: $CH_*(\mathcal{A}) := \bigoplus_{\substack{X_0,\dots,X_n \\ \mathrm{Ob}\,\mathcal{A}}} \hom(X_n, X_0) \otimes \hom(X_{n-1}, X_n) \otimes \cdots \otimes \hom(X_1, X_2) \otimes \hom(X_0, X_1)$

  where differentials are given by μ^n.



- Rmk: Let A be a commutative ring over a perfect field k.

  Then

  $$HH_n(A) \cong \Omega^n_{dR}(A)$$

  (Hochschild homology) ← algebraic de Rham forms

  And

  $H^*(S^1)$ - action gives the usual cochain complex

  $$\Omega^0 \to \Omega^1 \to \Omega^2 \to \dots$$

  (HKR Thm)

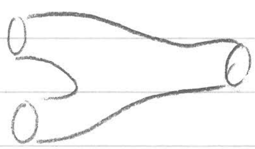
  ↳ Hochschild - Konstant - Rosenberg

- **Note:** Now let's talk about TFT's in 2-D

- **Recall:** Examples of $(\infty, 2)$-categories
  - Morita: obj = A_∞-alg
    hom = bimod
    2-hom = maps of bimods
  - A_∞Cat: obj = A_∞ cats w/ colims
    hom = colim-preserving functors
    2-hom = nat. trans

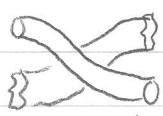
- **Recall:** A 2-D TFT is a symmetric monoidal functor $(Cob_2^{fr})^{\amalg} \to \mathcal{C}^{\otimes}$

- **Note:** New features in 2-D (If $Z(S^1) = V$)



pair of pants: $S^1 \amalg S^1 \to S^1$ $V \otimes V \xrightarrow{m} V$

Since Cob symmetric, there's equivalences

$X \amalg Y$  $Y \amalg X$

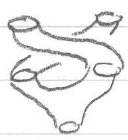
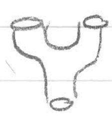
"swap"

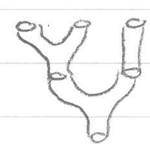
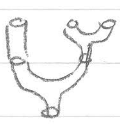
So (up to isotopy)

$$\begin{tikzcd} S^1 \amalg S^1 \arrow[dr, "pop"] \arrow[d, "swap"'] & \\ S^1 \amalg S^1 \arrow[r, "pop"'] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} V \otimes V \arrow[dr, "m"] \arrow[d, "swap"'] & \\ V \otimes V \arrow[r, "m"'] & V \end{tikzcd}$$

up to htpy

 $\overset{\text{diffeo}}{\cong}$ (in $\mathbb{R}^\infty$) 

Note:  vs.  $\rightsquigarrow$ Little 2-disks operad

Upshot: $Z(S^1)$ is an $\mathbb{E}_2$-algebra (Algebra over 2-disks operad)

- New Features (continued)

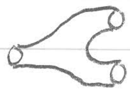
 $\longleftrightarrow$ $\mathbb{1} \longrightarrow V$

$\emptyset \xrightarrow{u} S^1$

Now 

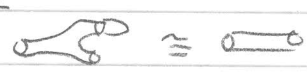
$m \circ u \cong id_{S^1}$ $\Longrightarrow$ $Z(u) : \mathbb{1} \to V$ is a unit for m

Also,

coproduct  , counit 

$\updownarrow$ $\updownarrow$

$\Delta : V \longrightarrow V \otimes V$ $\qquad \eta : V \longrightarrow \mathbb{1}$

with $\eta \circ \Delta = id$ by 

etc.

- Thm: Consider the $(\infty, 1)$-category $Cob^{fr}_{2,1}$ with

  objects $= \emptyset, S^1, S^1 \amalg S^1, \ldots, \amalg^k S^1, \ldots$

  homs $=$ cobordisms $X \longrightarrow Y$

  Then a symmetric $\otimes$ functor

  $$Cob_{2,1} \longrightarrow Vect_k^{\otimes}$$

  determines a commutative Frobenius algebra $/ k$

- Q: So when might we have a TFT

  $$Cob_2 \xrightarrow{Z} A_\infty Cat \ ?$$

- Note: From before, if $Z(pt) = A$,

  we know $Z(S^1) = CH_*(A)$

$$\begin{array}{ccc} & S^1 & \\ \emptyset_0 & \longrightarrow & \emptyset_0 \\ & \downarrow Z & \\ \mathbb{1}_{A_\infty Cat} & \longrightarrow & \mathbb{1}_{A_\infty Cat} \\ \text{\rotatebox{90}{$\cong$}} & & \\ Chain_{\mathbb{Z}} & \longrightarrow & Chain_{\mathbb{Z}} \end{array}$$

this is $- \otimes CH_*(A)$