Exercise: Let X be a ∞-grpd. Convince yourself $\mathrm{Hom}(x,x) \simeq \Omega_x X$

Last time: $\mathrm{Cob}_1^{fr} \to \mathrm{Morita}$

$pt \longrightarrow A$

$\bigcirc \longrightarrow HC_*(A) = A \otimes^L A.$

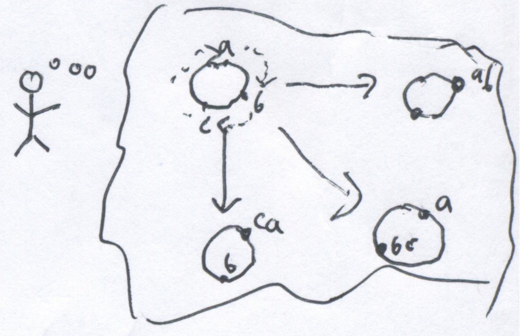
$HC_*(A) \simeq A \leftarrow A \otimes A \leftarrow \cdots$

$ab - ba \leftarrow a \otimes b$

$(-1)ba$

$ab \otimes c - a \otimes bc + ca \otimes b \leftarrow a \otimes b \otimes c$

some signs



Factorization homology / cosheaves on the ran space

Circle action on Hochschild homology

$Z: \mathrm{Cob}_{fr}^1 \to \mathcal{C}$

$Z(S^1)$ must have action of $\mathrm{Aut}(S^1)$.

We have $S^1 \in \mathrm{Hom}(\emptyset_0, \emptyset_0)$ so $\mathrm{Aut}(S^1) \simeq \Omega_{S^1} \mathrm{Hom}(\emptyset_0, \emptyset_0)$

∞-grpd by Exercise

A model for $\mathrm{Hom}(\emptyset_0, \emptyset_0) \simeq \{\text{compact 1-manifolds in } \mathbb{R}^\infty\}$

$\simeq \coprod_{[X]} \mathrm{Emb}(X, \mathbb{R}^\infty)/\mathrm{Diff}(X) \simeq \coprod_{[X]} */\mathrm{Diff}(X) \simeq \coprod B\mathrm{Diff}(X)$

Now clear that $\mathrm{Aut}(S^1) \simeq \mathrm{Diff}(S^1) \simeq S^1$

(Morally obvious - hard part is technical details of definitions as (∞,n)-categories)

Rmk Let $\mathcal{C}^\otimes = A_\infty \mathrm{Cat}$; objects A_∞ categories w/ colimits, morphisms colimit preserving functors

unique by Cobordism hypothesis

Now for a functor $Z: \mathrm{Cob}_1^{fr} \to A_\infty \mathrm{Cat}^\otimes$, $Z(S^1)$ is the Hochschild complex of A

$$CH_*(A) = \bigoplus_{x_0 \dots x_n \in \mathrm{Ob} A} \mathrm{Hom}(x_0, x_1) \otimes \cdots \otimes \mathrm{Hom}(x_{n-1}, x_n) \otimes \mathrm{Hom}(x_n, x_0)$$

If A is a comm. unital ring over a perfect field k

$H_*(CH) \simeq HH_*(A) \simeq \Omega^n_{dR}(A)$

But we want differential. $H^*(S^1)$ has an element of deg 1

So we get

$$\cdots \overset{2}{\Omega^2} \leftarrow \overset{1}{\Omega^1} \leftarrow \overset{0}{\Omega^0}$$

<u>2d TFT</u>: $Z : Cob^2_{fr} \to \mathcal{C}$. Examples of $\mathcal{C}$: A_∞ Cat ((∞,2)-cat) (remember colimit conditions)

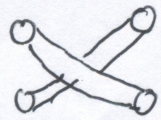
obj: A_∞-alg

Morita mor: bimodules

2mor: morphisms of bimodules

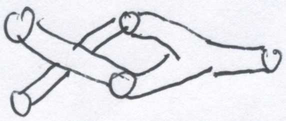
If $Z(S^1) = V$:

$m: V \otimes V \to V$

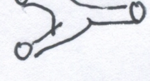


swap: $V \otimes V \to V \otimes V$

Note:



isotopic to



so ∃ homotopy from $m \circ swap$ and m

But this isotopy is not unique! $Z(S^1)$ is a E_2-algebra.

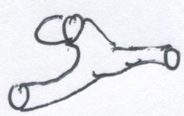
Little 2-disks operad:

space of maps $Z(S^1) \otimes Z(S^1) \otimes Z(S^1) \cdots \otimes Z(S^1) \to Z(S^1)$

↑ n copies

is space of n disks in a big disk.

unit: $u: 1 \to V$



we see $m \circ u \simeq id_V$.

Similarly, have coproduct, counit.. $V \xrightarrow{\Delta} V \otimes V$, $V \xrightarrow{counit} 1$

$counit \circ \Delta \simeq \Delta$

etc, etc..

<u>Thm</u> Consider the $(\infty,1)$-cat $Cob^{fr}_{2,1}$: $Hom(\emptyset, \emptyset)$ in Cob^{fr}_2

A symm. ⊗ functor $Cob_{2,1} \to Vect^{\otimes}_k$ determines a Frobenius algebra $(Z(S^1))$

When might we have a TFT $Cob_2 \xrightarrow{Z} A_\infty Cat$?

If $Z(pt) = A$, then $Z(S^1) \simeq CH_*(A)$

$\emptyset \longrightarrow \emptyset$

$Chain_k \xrightarrow[\otimes CH_*(A)]{} Chain_k$