- Today: Start discussion of Calabi-Yau categories
  (see Costello, "Top. conformal field theory and CY categories")

- Defn: $\mathcal{O}$ = category of open strings

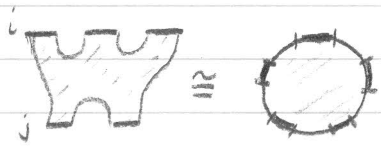
  where
  - $\mathrm{Ob}\,\mathcal{O} \cong \mathbb{Z}_{\geq 0}$

    We'll draw an element $i \in \mathrm{ob}\,\mathcal{O}$ as i disjoint intervals

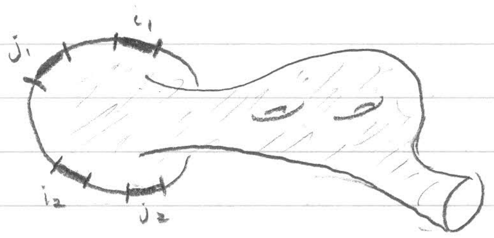
    e.g. — — — — —

  - $\hom(i,j)$ (a space) $= \left\{ \begin{array}{l} \text{compact Riemann surfaces w/ } \partial\text{, say } \Sigma\text{, together} \\ \text{w/ embedding } \phi : I^{\amalg i} \amalg I^{\amalg j} \hookrightarrow \partial\Sigma \\ \text{s.t. } \phi|_{I^{\amalg i}} \text{ preserves orientation} \\ \quad\;\; \phi|_{I^{\amalg j}} \text{ reverses orientation} \end{array} \right\}$

  For example:



or

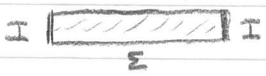


- Note: Consider $\mathcal{D} \subset \mathcal{O}$ where all components are either disks or annuli (of certain type).
- Rmk: $\mathcal{D}, \mathcal{O}$ are symmetric monoidal categories via $\amalg$.

- Q: What does a symm. mon. functor $\mathcal{D} \to \mathrm{Chain}_{\mathbb{Q}}^{\otimes}$ look like?

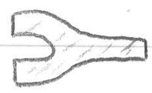
  (1) $\emptyset \longmapsto \mathbb{Q}$
  (2) $I \longmapsto V$
  (3) $I^{\amalg j} \longmapsto V^{\otimes j}$

• **Note**: On morphisms (considering only disks for now)



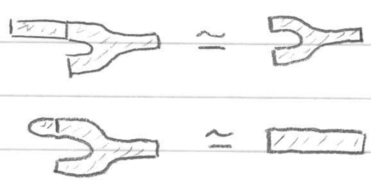
$\mapsto (V \to V)$

$\mapsto (\mathbb{Q} \to V)$



$\mapsto (V \otimes V \to V)$

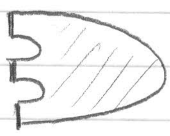
Compositions



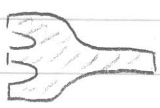
(left = in, right = out)

So far: Looks like choice of $V \in \mathrm{Chain}_{\mathbb{Q}}$ and some algebraic structure on V

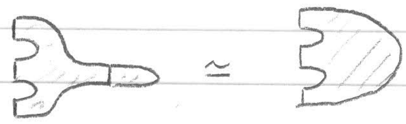
$V \xrightarrow{tr} \mathbb{Q}$



$V^{\otimes i} \to \mathbb{Q}$



$V^{\otimes i} \xrightarrow[m_i]{} V$



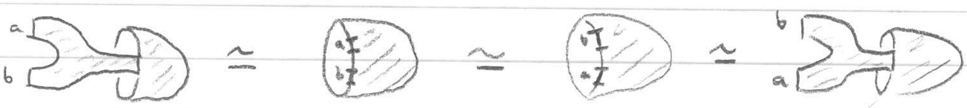
$V^{\otimes i} \xrightarrow{m_i} V \xrightarrow{tr} \mathbb{Q}$

Rmk: Another model for $\mathcal{O}, \mathcal{D}$ is to use marked pts, e.g.

The trace map has cyclic symmetry

$$tr(a \cdot b) = tr(b \cdot a)$$

Why?



Get a map
$$V \longrightarrow V^\vee$$
$$a \longmapsto (\bullet \mapsto tr(a \bullet))$$

• "Defn": A Calabi-Yau algebra of dimension 0 is an $\underbrace{\text{associative algebra}}$ V with $tr: V \to \mathbb{Q}$

↳ Could be dg algebra or A_∞-algebra

• **Note**: Let's do a many object version (next page)

- **Note**: Fix Λ a set of "branes" (think: Lagrangians)

  Define: $\mathcal{O}_\Lambda$ to be a category whose objects are disjoint unions of $(\underset{\text{interval}}{I}, \underbrace{s,t}_{\text{source + target}})$, $s,t \in \Lambda$

  i.e. disjoint unions of

$\hom\big((I_i, s_i, t_i)_i \,,\, (I_j, s_j, t_j)_j \big)$

|| space

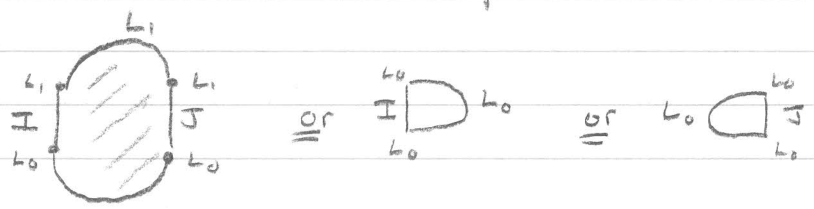
$\{ (\Sigma, \text{labels}, \phi) \mid \Sigma, \phi$ as before

labels are on components of $\partial\Sigma \setminus \text{image}(\phi)$

labelled by elements of Λ

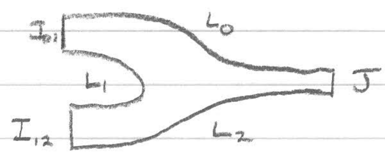
compatible w/ labels on $I_i, I_j \}$

e.g.



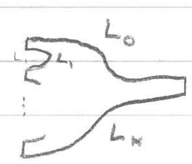
- **Q**: What's a functor $\mathcal{O}_\Lambda^{\sqcup} \to \mathrm{Chain}_{\mathbb{Q}}^{\otimes}$?

  Well:
  - To every $L_s, L_t \in \Lambda$, a chain complex $\hom(L_s, L_t)$
  - To every



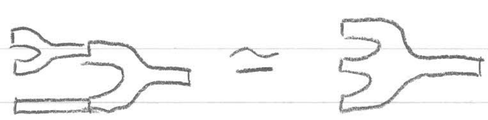
    a map $\hom(L_0, L_1) \otimes \hom(L_1, L_2) \xrightarrow{\mu^2} \hom(L_0, L_2)$

  - More generally

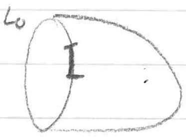


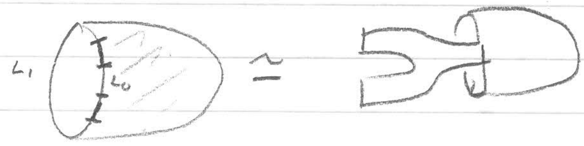
    $\rightsquigarrow \quad \mu^k$

  - Relations:



(continued...)

• Also  $\rightsquigarrow$ $\hom(L_0, L_0) \xrightarrow{tr} k \quad \forall L_0 \in \Lambda$



$$\begin{array}{ccc} \hom(L_0,L_1) \otimes \hom(L_1,L_0) & \longrightarrow & k \quad \simeq tr \circ \mu^2 \\ \downarrow \text{swap} \quad \circlearrowleft & \nearrow & \\ \hom(L_1,L_0) \otimes \hom(L_0,L_1) & & \end{array}$$

• Defn: A Calabi-Yau category of dimension zero is an (A_∞) category together with a map

$$CC_*(\text{category}) \longrightarrow k$$
(Hochschild chains)

with a certain non-degeneracy condition

• Note: Giving a map $\hom(L_0, L_0) \longrightarrow k \quad \forall L_0$ the same as giving a map

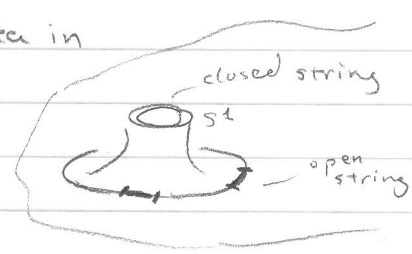
$$\bigoplus_{L_0 \in \Lambda} \hom(L_0, L_0) \longrightarrow k$$

The cyclic invariance means

$$\begin{array}{ccc} \bigoplus \hom(L_d, L_0) \otimes \cdots \otimes \hom(L_0, L_1) & & \\ \downarrow \mu & \searrow & \\ \hom(L_0,L_0) & \xrightarrow{tr} & k \end{array}$$

so $CC_* \to k$

• Rmk: Nowadays, people may ask for more data in defn of CY category.



• Defn: Let $\mathcal{OC}$ be the category whose objects are S^1, $\emptyset$, interval, and $\amalg$ of them (if you want to incorporate Λ, don't label the S^1).

A morphism is Σ with $\phi: I^{\amalg i} \amalg I^{\amalg j} \amalg S^{1\,\amalg i'} \amalg S^{1\,\amalg j'} \longrightarrow \partial\Sigma$ (with compatible Λ labellings) (some constant, when target/source of Σ are $\emptyset$)

• Q: $\mathcal{O} \longrightarrow \mathcal{OC} \dashrightarrow \text{Chain}_\otimes$ $\rightarrow$ does this exist?