Hiro. 10/30

<u>Today</u> Start discussion of CY cat.

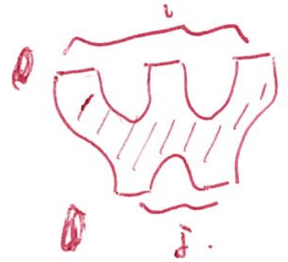
(Costello, TCFT & CY cat.)

<u>Def.</u> $\mathcal{O}$. (cat. of open strings)

$\mathrm{Ob}\,\mathcal{O} \simeq \mathbb{Z}_{\geq 0}$. $\underbrace{\underline{I}\ \underline{I}\ \underline{I}\ \underline{I}}_{i} \equiv i$.

$\hom(i, j) = \{$ Rie. sf. Σ together w/ embedding $\phi : I^{\amalg i} \amalg I^{\amalg j} \to \partial\Sigma \}$
s.t. $\phi|_{I^{\amalg i}}$ preserves orientation, $\phi|_{I^{\amalg j}}$ reverses.

e.g.



Consider $\mathcal{D} \subset \mathcal{O}$ where all conn. comp. are either disks or annuli.

$\mathcal{D}, \mathcal{O}$ are sym. monoidal cat. via $\amalg$.

What does a sym. mon. functor $\mathcal{D} \longrightarrow \mathrm{Chain}^{\otimes}_{\mathbb{Q}}$ look like?

1) $\phi \longmapsto \mathbb{Q}$.
2) $I \longmapsto V$
3) $I^{\amalg j} \longmapsto V^{\otimes j}$.

On morphisms (consider only disks for now):

 $\longmapsto$ $V \to V$.

 $\longmapsto$ $\mathbb{Q} \to V$

 $\longmapsto$ $\begin{matrix} V \\ \otimes \\ V \end{matrix} \to V$

<u>Composition.</u>

So far, looks like a choice of $V \in \mathrm{Chain}_{\mathbb{Q}}$ and some alg str. on V.

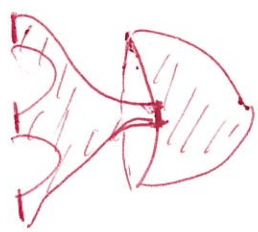
$\longmapsto \quad V \xrightarrow{\mathrm{tr.}} \mathbb{Q}$

$I \longrightarrow \phi$

Rmk. Another model for $\mathcal{O}, \mathcal{D}$ is to use marked pts & allowing nodal bdy.

$\longmapsto \quad V^{\otimes i} \longrightarrow \mathbb{Q}$

$I^{\amalg i} \longrightarrow \phi$.



$\longmapsto \quad V^{\otimes i} \xrightarrow{m} V \xrightarrow{\mathrm{tr.}} \mathbb{Q}$

$I^{\amalg i} \to I \to \phi$

This tr. map has a cyclic sym: $\mathrm{tr}(a \cdot b) = \mathrm{tr}(b \cdot a)$

$\rightsquigarrow \quad V \longrightarrow V^\vee$

$a \longmapsto (\bullet \longmapsto \mathrm{tr}(a \bullet))$

"<u>Def</u>" A <u>CY alg. of dim. 0</u> is an assoc. alg. V w/ $\mathrm{tr}: V \to \mathbb{Q}$ which is cyclically invar., $V \longrightarrow V^\vee$ is isom.

(could be dg alg. or A_∞ alg)

Many obj. version:

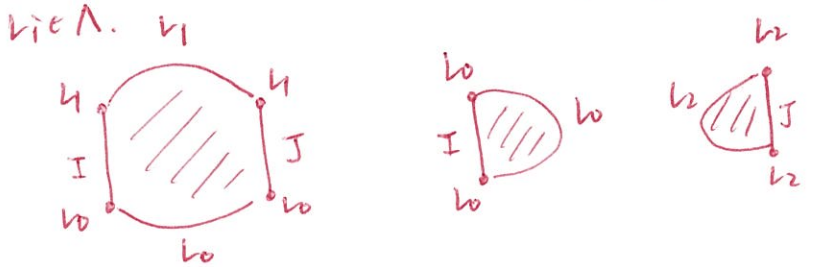
Fix Λ: a set of "branes". (think of Lag.)

Define $\mathcal{O}_\Lambda$ to be a cat. whose obj. are disjoint unions of (I, s, t) — Interval; $s, t \in \Lambda$

$\hom\big((I_i, s_i, t_i)_i, (J_j, s_j, t_j)_j\big)$

$= \{ (\Sigma, \text{labels}, \phi) \mid \Sigma, \phi$ as before, labels on comp. of $\partial\Sigma \setminus \mathrm{Im}(\phi)$. labelled by elts of Λ, compatibly w/ labels on I_i, J_j. $\}$

$L_i \in \Lambda$.

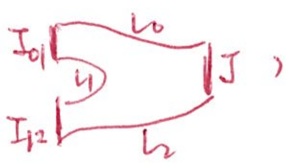


What's a functor $\mathcal{D}_\Lambda \longrightarrow \mathrm{Chain}_\otimes$?

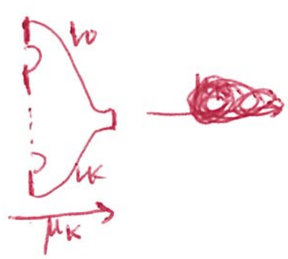
$\amalg \longmapsto \otimes$

To every $L_s, L_t \in \Lambda$, a chain cpx $\hom(L_s, L_t)$.

To every



, $\hom(L_0, L_1) \otimes \hom(L_1, L_2) \longrightarrow \hom(L_0, L_2)$.



$\hom(L_0, L_0) \xrightarrow{tr} k$. $\forall L_0 \in \Lambda$.

$$\begin{tikzcd} \hom(L_0,L_1) \otimes \hom(L_1,L_0) \arrow[r] \arrow[d, "\text{swap}"] \arrow[dr, phantom, "\circlearrowright H"] & k \\ \hom(L_1,L_0) \otimes \hom(L_0,L_1) \arrow[ur] & \end{tikzcd}$$

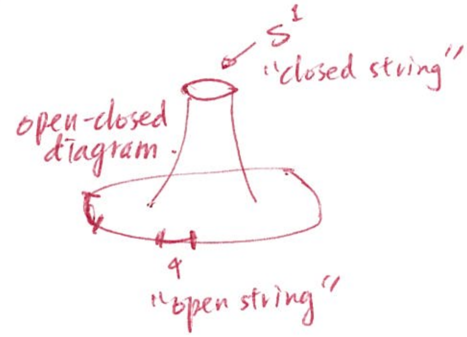
<u>Note</u>: Giving a map $\hom(L_0, L_0) \xrightarrow{tr} k$ $\forall L_0$ is the same thing as giving a map $\bigoplus_{L_0 \in \Lambda} \hom(L_0, L_0) \longrightarrow k$.

The cyclic invar. means

$$\begin{tikzcd} \bigoplus \hom(L_0,L_1) \otimes \cdots \otimes \hom(L_k,L_0) \arrow[d, "\mu"] \arrow[dr] \arrow[dr, phantom, "\circlearrowright H"] & \\ \hom(L_0,L_0) \arrow[r, "tr"] & k \end{tikzcd}$$

<u>Def</u> A <u>CY cat. of dim. 0</u>. is an (A_∞) cat. together w/ a map. $CC_*(\text{cat.}) \longrightarrow \mathbb{k}$. w/ a non-degeneracy condition.

<u>Rmk</u>: Now-a-days. people may ask for more data.



<u>Def</u> Let $\mathcal{OC}$ be the cat. whose obj. are S^1, ϕ. interval. and $\amalg$ of them.

(— — — ⊖ ⊖)

(If you want to incorporate Λ, don't label $\overset{S^1}{\circ}$)

A morphism is a Σ w/ $\phi : I^{\amalg i} \amalg I^{\amalg j} \amalg (S^1)^{\amalg i'} \amalg (S^1)^{\amalg j'} \hookrightarrow \partial \Sigma$

(w/ Λ, labellings must be compatible).

(Some constraint when target. source are ϕ)

<u>Q</u>: Given

$$\begin{tikzcd} \mathcal{O} \arrow[r, "F"] \arrow[d] & \mathrm{Chain}_{\mathbb{Q}} . \\ \mathcal{OC} \arrow[ur, dashed, "\exists ?"'] & \end{tikzcd}$$