i.>

277 (Hiro) - 2/11

__Exer.__ If $Z : Cob_1^{fr} \to Chain_k^{\otimes}$

is a TFT. Show

$$\bigoplus_{i \in \mathbb{Z}} H^i(Z(pt)) \in k\text{-Mod}$$

has finite dimension over k (k is a __field__). __Comment__ Can also work with perfect complexes

__Warm-up:__

__Prop.__ Let

$$Z : Cob_1^{fr} \to Vect_k^{\otimes}$$

be a TFT. Then

- $Z(pt) = V$ is finite dim.
- $Z(pt^-)$ is equipped with an

point with negative orientation

linear dual

$\cong$ to $V^\vee = Hom(V, k)$.

Let's write a quick proof.

<u>Lemma.</u> TFAE

(1) $V^\vee \otimes V \to \hom(V,V)$

$\phi \otimes u \mapsto f: v \mapsto \phi(v)u$

is an iso.

(2) $V^\vee \otimes V \to \hom(V,V)$

has id_V in it's image.

(3) $\dim_k V < \infty$.

<u>Proof of proposition.</u> Consider $W := Z(pt^-)$,

(Zorro's lemma)

Call

$Z(\supset) = f: V \otimes W \to k$

$Z(\subset) = g: k \to W \otimes V$

Note f induces a map

$$f^{ad}: W \to V^\vee$$

$$w \mapsto f(- \otimes w)$$

Consider the composition

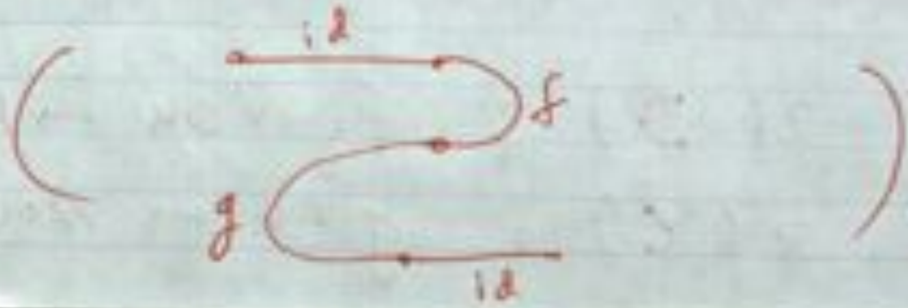
$$k \xrightarrow{g} W \otimes V \xrightarrow{f^{ad} \otimes id} V^\vee \otimes V \to End(V)$$

If we can show it hits id we are done. Let's explore what it means

$$1 \mapsto \sum w_i \otimes u_i + \dots \mapsto$$

$$\underbrace{\left(\vec{v} \mapsto \sum f(\vec{v} \otimes \vec{w}_i)\, \vec{u}_i\right)}_{\text{this map \underline{is}}}$$

z



why? because this coboralism is

$$= \begin{matrix} V \\ \otimes \\ 1 \end{matrix} \xmapsto{\mathrm{id}\otimes g} \begin{matrix} V \\ \otimes \\ \sum w_i \otimes v_i \end{matrix} = \sum_i \begin{matrix} v \\ \otimes \\ w_i \\ \otimes \\ v_i \end{matrix} \xmapsto{f \otimes \mathrm{id}}$$

$$\sum f(v \otimes w_i) v_i.$$

~~So this is a result of Zorn's lemma,~~

By lemma, $\dim V < \infty$.

(2) $f: V \otimes W \to k$

induces a map $f^{ad}: W \to V^\vee$

since

$$W \otimes V \xrightarrow{f^{ad} \otimes \mathrm{id}} V^\vee \otimes V \to \mathrm{End}(V)$$

hits id_V, f^{ad} must be <u>surjection</u>.

By considering , we get

a map $V \to W^\vee$, also a surjection. $\blacksquare$

⚠ Rmk. The TFT does not produce an iso $\cong$ between V to $V^\vee$; merely a choice of W and a morphism $W \xrightarrow{\cong} V^\vee$.

The exercise is basically to repeat this for chain-complexes.

A similar picture arises in considering CY algebras/categories.

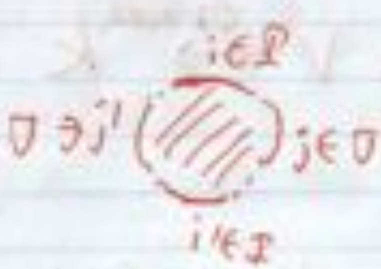
Last time. Consider a category $\mathcal{D}$ whose objects are $\amalg$ intervals, i.e.

ϕ, ⟝⟞, ⟝⟞ ⟝⟞, ...

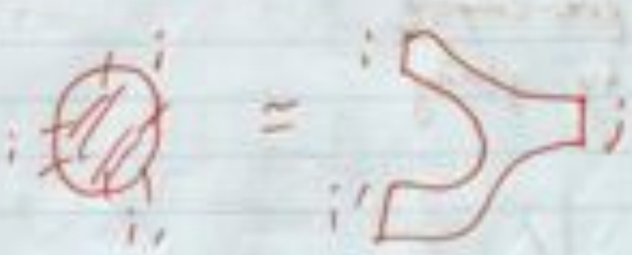
$$\mathrm{Hom}(\underset{\text{intervals}}{I, J}) = \left\{ \begin{array}{l} \text{Rieman surface } \Sigma, \\ \text{cpt wrt } \partial, \text{ w/ embedding} \\ I \sqcup J \to \partial\Sigma \end{array} \right\}$$

Today require that each component of Σ is a disc.

Ex.



Ex.



D has symmetric monoidal $\otimes$ of $\sqcup$ and we consider sym $\otimes$ functors

$$D \longrightarrow \mathrm{Chain}_k^{\otimes}$$

$$\emptyset \mapsto k$$

$$I \mapsto V$$

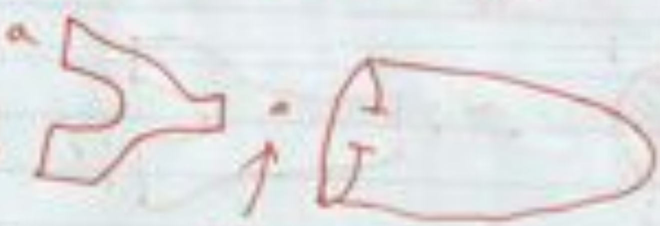
lots of math structure (but contractible choice!)

 $\mapsto V \otimes V \xrightarrow{m} V$

one incoming?

( = ) $\mapsto V \xrightarrow{\text{trace}} k$

This tr has cyclic symmetry:

 $= \mathrm{tr}(m(a,b))$

pre-compose by gluing

$\wr$

 rotate $\simeq$ 

$\simeq$ 

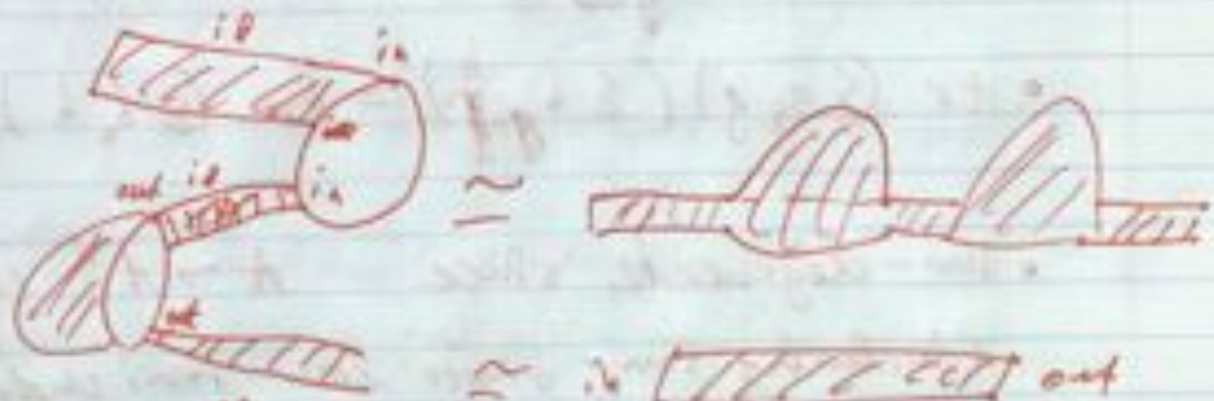
$= tr(m(b,a))$

<u>Def.</u> A <u>CY-alg</u> of dim 0 is an algebra A equipped w/ cyclically inv $tr: A \to k$ s.t. the induced map

$$A \to A^\vee$$
$$a \mapsto tr(a\,-)$$

is an equivalence (i.e., quasi-iso of chain complexes).

<u>Rmk.</u> consider "Zorro's lemma for R1":



So this picture shows that any functor from the cat D (sym, monoidal) determines CY-alg of dim 0 (this is similar to our warm-up).

This is kind of a natural thing to think about if you are interested in string theory/Fukaya categories.

<u>Ex.</u> fix G finite group, k a field. Set $A = k[G]$. Define

$$tr : A \longrightarrow k$$
$$\sum_{g\in G} a_g \cdot g \mapsto a_e$$

- $tr\,(\sum a_g g)(\sum b_h h) = tr\,(\sum b_h h)(\sum a_g g)$
- non-degenerate, since $A \to A^\vee$ is an iso.

(Wedderburg thm - finite dim semi simple alg —)

307

Ex. $A = M_{n\times n}(k)$.

$tr : A \to k$

$X \mapsto tr(X)$

- $tr(XY) = tr(YX)$
- $A \to A^\vee$ non-deg.

with a "twist"

Ex. Fix X a cpt, oriented, n-dim mfld.

Let $A = \Omega_{dR}(X;R)$

Now define a trace

$$tr : \Omega_{dR}(X;R) \to R,$$
$$\alpha \mapsto \int_{[X]} \alpha.$$

- $tr(\alpha \wedge \beta) = (-1)^{|\beta||\alpha|} tr(\beta \wedge \alpha)$.
- Non-deg by P.D. (But anxious about $A^\vee$, either

(1) use A is an elliptic complex

(2) take $A = H^*_{dR}(X;R)$).

This is NOT a CY-alg of $\overset{n?}{\dim}$ 0

(Tam tam tam!!! Surprise!)

First of all,

(1) tr is NOT a map

$$\Omega^\bullet \to k$$

of deg zero. It has $\boxed{\text{deg} = -n.}$

should have written: $\boxed{\Omega^\bullet \xrightarrow{tr} k[-n]}$

is a deg zero map.

And the induced map

$$\boxed{A \to A^\vee[-n]}$$

is an equiv.

<u>Def.</u> A CY-alg of dim d is an (dg, A_∞) alg A w/a cyclically inv $tr: A \to k[-d]$ and

$$A \to A^\vee[-d] \text{ is an equiv.}$$

Our frustration. We had a beautiful category D which captures the notion of CY-algs, but

$t \to k$

must be $\deg = 0$, so we can't account for shifts. Also, this has not a lot to do with Fukaya categories.