Math 282 (4/11)

Last time. We saw example of CY-alg where $\dim \neq 0$!

($A = H^{\cdot}_{dR}(X)$, X compact or. $\mathrm{tr} : A \to k[+d]$, $\alpha \mapsto \int_{[X]} \alpha$)

Q. Can we define a version of D that incorporates degree shift? If so, call it D^d.

Def. A CY-alg of $\dim = d$ is symm $\otimes$ functor $D^d \to$ Chains.

A: give well chosen local systems to each mon-space.

Def. Fix X a top. space. A local system (of cochain cpx) is a functor $X \to$ Chains.

Digression. To make you happy, short intro to ∞-cat. (A particular model of $(\infty,1)$-categories).

Def. Let Δ be the cat w/ objects = finite, linearly ordered non-empty sets

Ex.

$$\mathrm{ob}\,\Delta \supset \{ \underset{\{0\}}{\underset{\uparrow}{[0]}},\ \underset{\{0<1\}}{\underset{\uparrow}{[1]}},\ \underset{\{0<1<2\}}{\underset{\uparrow}{[2]}},\ \ldots \}$$

$$\mathrm{hom}(I,J) \overset{\text{set}}{=} \left\{ \begin{matrix} \text{poset map} \\ I \to J \end{matrix} \right\}$$

, i.e., let $f: I \to J$ s.t. $i \le i'$ iff $f(i) \le f(i')$. That is just weakly order preserving maps.

Def. A simplicial set is a functor

$$\Delta^{op} \to Sets$$
$$[i] \mapsto X_i$$

~~Ex. Let X be a topological space~~

Ex. Fix $n \geq 0$. We consider

$$\Delta^n := \hom_{\Delta}(-, [n]) : \Delta^{op} \to Sets$$

that's a simplicial set!

Ex. Fix X a space. Then define

$$\Delta^{op} \to Sets$$
$$[n] \mapsto \mathrm{Hom}_{spaces}(\Delta^n, X)$$

space

i.e. continuous functions

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What does it do to morphism?

Note $\exists$ functor

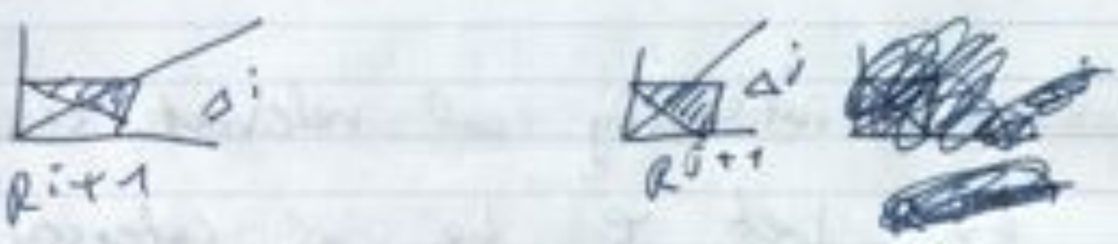
$$\Delta \longrightarrow \text{Spaces}$$
$$[i] \longmapsto \Delta^i = \{(t_0,\dots,t_i) \mid \textstyle\sum t_j = 1, \ t_j \geq 0\}$$
$$\cap$$
$$\mathbb{R}^{i+1}$$

Given a poset map $\rho \to \sigma$, there is an induced map $\Delta^\rho \longrightarrow \Delta^\sigma$.

$(t_0,\dots,t_\rho) \mapsto (s_0,\dots,s_\sigma)$ where $s_j = \sum_{i \in \sigma^{-1}(j)} t_i$

<u>Ex.</u> If $\rho = [n]$
$\sigma = [n+1]$

then $\rho \xrightarrow{\partial_j} \sigma$ is "skip a element $j \in \sigma$". What's going on? Picturally



This induces the inclusion

$$\Delta^i \hookrightarrow \Delta^{j+1}$$

of the j^{th} face.

So compose with the functor

$$\Delta^{op} \xrightarrow{\Delta} \text{Spaces}^{op} \xrightarrow[\hom(-,\Delta)]{} \text{Set}$$

You've seen this before!

Ex. $C_*^{sing}(X)$ is obtained by taking a free abelian group generated by each x_i, then defining the i^{th} differential as the alternating sum of ∂_j's.

More interesting and relevant example.

<u>Ex</u> Let $\mathcal{C}$ be a category.

This defines a functor

$$\Delta^{op} \xrightarrow{N(\mathcal{C})} Set,$$

$$[i] \mapsto \mathrm{Fun}([i], \mathcal{C})$$

Note that any poset is a category.

Elements of poset = objs of category

$$\hom(p,q) = \begin{cases} * & , p \le q \\ \emptyset & , \text{else} \end{cases}$$

Then

$$N(\mathcal{C})_0 = \mathrm{Fun}([0], \mathcal{C}) \underset{Set}{\cong} \mathrm{ob}\, \mathcal{C}.$$

cat w/ one object "0", and unique morphism $0 \to 0$

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Idea.

~~so~~ every category gives an example of simplicial set ...

$$N(\mathcal{C})_1 = \mathrm{Fun}([1], \mathcal{C})$$

cat w/ obj "0", "1" and $\overset{id}{\circlearrowleft} 0 \longrightarrow 1 \overset{id}{\circlearrowright}$

$$\underset{set}{\cong} \coprod_{\substack{x_0, x_1 \\ \in ob(\mathcal{C})}} \hom_{\mathcal{C}}(x_0, x_1)$$

What about morphism? consider

$$[0] \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} [1]$$

$$\begin{cases} \partial_0(0) = 1 \\ \partial_1(0) = 0 \end{cases}$$

give $\quad ob(\mathcal{C}) = N(\mathcal{C})_0 \underset{\partial_1 = source}{\overset{\partial_0 = target}{\leftleftarrows}} N(\mathcal{C})_1 = \mathrm{Morphism}(\mathcal{C})$

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let's figure

$$N(\mathcal{C})_2 = \mathrm{Fun}([2], \mathcal{C})$$

$$\cong \left\{ \begin{tikzcd} x_0 \arrow[rr, "f_{02}"] \arrow[dr, "f_{01}"'] & \circlearrowleft & x_2 \\ & x_1 \arrow[ur, "f_{12}"'] & \end{tikzcd} \right\}$$

commutative triangle at $\mathcal{C}$

and $[1] \underset{d_2}{\overset{d_0}{\underset{d_1}{\rightrightarrows}}} [2]$ correspond to

3 maps $N(\mathcal{C})_1 \leftleftarrows\!\leftarrow N(\mathcal{C})_2$ where they act

by

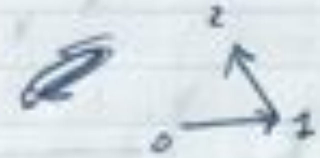
$$f_{12} \overset{d_0}{\longmapsfrom} \begin{tikzcd} \cdot \arrow[r, "f_{02}"] \arrow[d, "f_{01}"'] & \cdot \\ \cdot \arrow[ur, "f_{12}"'] & \end{tikzcd}$$

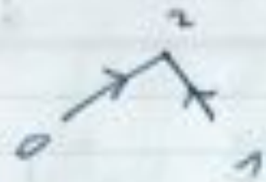
$$f_{02} \overset{d_1}{\longmapsfrom}$$

$$f_{01} \overset{d_2}{\longmapsfrom}$$

Defn. Let $\Lambda^n_k \subset \Delta^n$, $0 \le k \le n$ be the set obtained by removing the k^{th} face (opposite vertex k)

Ex. $\Lambda^2_0 =$ 

$\Lambda^2_1 =$ 

$\Lambda^2_2 =$ 

Prop. Fix $\mathcal{C}$. Then for every diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \qquad (0<i<n)$$

$\exists!$ f making this diagram commute.

<u>Ex.</u> $n=2$, $i=1$. A functor

$$\Lambda^2_1 \longrightarrow \mathcal{C}$$

is a choice of

$$\begin{tikzcd} & X_2 & \\ X_0 \arrow[rr, "f_{01}"'] & & X_1 \arrow[ul, "f_{12}"'] \end{tikzcd}$$

$\rightsquigarrow$ $\exists!$ filler (composition)

i.e.,

$$\begin{tikzcd} & X_2 \\ X_0 \arrow[ur, "f_{12}\circ f_{01}"] \arrow[r] \arrow[ur, phantom, shift right=3, "\circlearrowleft"] & X_1 \arrow[u, "f_{12}"'] \end{tikzcd}$$

is a commutative triangle.

<u>Thm.</u>

$$\left\{ \begin{array}{c} \text{sSets satisfying} \\ \text{our Prop} \end{array} \right\} \underset{\substack{\text{equiv}\\ \text{of}\\ \text{cat}}}{\simeq} \left\{ \begin{array}{c} \text{small} \\ \text{categories} \end{array} \right\}$$

$\cap$ subCat

sSets

__maps:__

$$N(\mathcal{C}) \longleftarrow\!\shortmid \ \mathcal{C}$$

$$X \longmapsto \mathcal{C}_X := \left\{ \begin{array}{l} \mathrm{ob} = X_0 \\ \hom(x,y) \subset X_1 \\ \text{whose face} \\ \text{maps go to} \\ x, y \end{array} \right\}$$

__Exer__ $n=3$, $i=1,2 \Rightarrow$ shows composition is associative.

So the condition of filling inner horns looks a lot like the property of being a category. What happens if we remove the condition of __unique__ filling?

Def. A simplicial set $\mathcal{C}$ is called an $\underline{\infty\text{-cat}}$ if $\forall$ diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "f"] & \end{tikzcd} \qquad 0<i<n$$

$\exists f$ making diagram commute.

Why nice?

(1) Thm. If $\mathcal{C}, \mathcal{D}$ are ∞-cat, then

$$\mathrm{Fun}(\mathcal{C},\mathcal{D}) := \hom_{sSet}(\mathcal{C},\mathcal{D})$$
(natural trans)

this is an ∞-cat.

The RHS

$$\hom_{\text{sSets}}(\mathcal{C}, D)$$

is a sSet.

$\hom(\mathcal{C}, D)_i := \text{NaturalTrans}(\mathcal{C} \times \underline{\Delta}^i, D)$

This existed like 20 years before Joyal said "let's use it to define categories". Very classical.

In almost every other model, this ~~that~~ Then is very hard to prove

(2) Great Defn of (homotopy) (co) limits.

(3) Amazing Grothendieck constructions

Probably the best thing in ∞-cat!

Classical. Given a functor

$\mathcal{C} \xrightarrow{f} \mathrm{Sets}$

$\exists$(! up to equiv) a category

$$\begin{array}{c} \mathcal{C}_F \\ \downarrow \\ \mathcal{C} \end{array}$$

this is kind of "classifying space construction". Think G-torsors. Lurie HTT calls it "(un) straightening".

<u>Other models.</u> (just in name)

- complete Segal spaces ("simplical spaces")
- Top-enriched cat
- Cat enriched in simplicial sets
- simplicial Model cat

The best thing about Segal spaces is the generalization to (∞, n), which is immediate and hard/impossible for $(\infty,1)$-cat in Lurie sense.

<u>One bad thing</u> about the Lurie model ... — where id come from?

the unit

$$N(\mathcal{C})_1 \longleftarrow N(\mathcal{C})_0$$

$$id_X \longmapsto X$$

collapse all $\quad [1] \longrightarrow [0]$

$$\mathcal{C}_1 \leftarrow \mathcal{C}_0$$

in Lurie's model <u>we have</u> to <u>specify</u> unit for each $X \in ob(\mathcal{C}) = \mathcal{C}_0$

The fact you have to choose is the only short coming of this approach.