LECTURE 28

# Introduction to $\infty$-categories

Though I prefer hand-written notes, I'm typing this up in case it's a helpful resource.

I've been agnostic in this class about what I mean when I say "$(\infty, 1)$-category." To put grumblings to rest, I'll just specify a single model of $(\infty, 1)$-category that I like, called $\infty$-categories. This terminology is due to Lurie—synonyms include quasi-categories and weak Kan complexes.

Just to give you a road map: An $(\infty, 1)$-category is supposed to capture two ideas at once:

(1) The idea that two objects can have a *space*, and not just a set, of morphisms between them. A good example is the category of topological spaces—here, the set of continuous functions from $X$ to $Y$ can actually be given a topology.
(2) Sometimes, composition of morphisms is not uniquely defined, and associativity of composition does not hold on the nose, but only up to homotopy. We've seen an example in this class via Fukaya categories. Another example is the category of points on $X$ whose morphisms are curves on $X$.

This second idea (2) is usually the one least palpable to newcomers. Don't be afraid, as you've seen an instance of this before: The fundamental group. We are content to define $\pi_1(X, x)$ by simply declaring two loops to be equivalent if they are homotopic. The main motivation behind imposing such an equivalence relation is to make composition of loops associative. The drawback, of course, is that you've modded out by homotopies—modding out, by its very nature, loses information. One can instead preserve this information, but at the cost of living in a world where associativity need not hold on the nose, and only up to homotopy.

In my class, I use the term $(\infty, 1)$-category to mean anything that captures these two notions. However, when I say "$\infty$-category," I mean the very specific model that I give in Definition 28.18. Yes, the only difference is the parentheses, and the number 1.

**Remark 28.1.** Unless otherwise stated, when I say "category," I really mean category, in the sense of Mac Lane. So the three terms

category $\quad \infty$-category $\quad (\infty, 1)$-category

all mean different things in this note. I will not mathematically define the last term.

**Remark 28.2.** This is a long document. I'm amazed I covered all this in one hour.

**Remark 28.3.** The content of these notes are probably equivalent to 1.1.2 of Lurie's Higher Topos Theory. However, the theory of $\infty$-categories was first developed by Joyal under the name *quasi-categories*.

## 1. Preliminaries

### 1.1. The category $\Delta$.

**Definition 28.4.** Let $\Delta$ be the category of non-empty finite, linearly ordered sets. To be precise:

(1) An object of $\Delta$ is a finite, non-empty set with a linear poset structure. An example is the set $\{0, 1, \ldots, n\}$ with the ordering

$$0 \leq 1 \leq 2 \leq \ldots \leq n.$$

This particular object is denoted $[n]$. As an example, $[0]$ is a poset with one element.

(2) A morphism from $I$ to $J$ is a map of posets. That is, a function $f : I \to J$ such that

$$i \leq i' \implies f(i) \leq f(i').$$

Such an $f$ is also called a weakly order-preserving map.

**Remark 28.5.** Any object of $\Delta$ is uniquely isomorphic to the object $[n]$ for some $n$. So the functor out of $\Delta$ is determined, up to natural isomorphism, by what it does on the $[n]$.

**Example 28.6.** The most important kinds of morphisms are the following two:

- The injective maps $\delta_i : [n] \to [n+1]$ skipping the element $i \in [n+1]$. There are $(n+1)$ of these. These are called fact maps.
- The surjective maps $\sigma_i : [n] \to [n-1]$ which send $i$ and $i+1 \in [n]$ to the same element, $i \in [n-1]$. There are $(n-1)$ of these. These are called degeneracy maps.

You can check that any morphism $f : [n] \to [m]$ can be factored as a composition of these basic kinds of maps.

**Remark 28.7.** As we'll see, this category is a simple way of capturing the combinatorics of $k$-simplices for all $k$ at once. For a mysterious reason, this category has proven ridiculously useful in all kinds of topology—not just for the theory of higher categories.

**1.2. Simplicial sets.** Every example in this section is important, and will be used immediately.

**Definition 28.8.** A simplicial set is a functor

$$\Delta^{\mathrm{op}} \to \mathsf{Sets}.$$

Given a simplicial set $X$, we will denote $X([i])$ by $X_i$. The functions induced by $\delta_i, \sigma_i$ will be denoted $d_i, s_i$, respectively.

A *map*, or *morphism*, from one simplicial set to another is a natural transformation.

**Example 28.9** (Simplices)**.** A basic example is the representable functor

$$\underline{\Delta}^n : \Delta^{\mathrm{op}} \to \mathsf{Sets}, \qquad [i] \mapsto \hom_\Delta([i], [n]).$$

As an example, $\underline{\Delta}^0$ is the constant functor: It sends every $[i]$ to a one-element set.

$\underline{\Delta}^1$ is more interesting: It sends $[0]$ to a two-element set. $[1]$ is sent to a three-element set. How are these two sets related in terms of the maps $\delta_i$ and $s_i$ from above?

Keep going for $\underline{\Delta}^n$. Convince yourself that there is a single element of $\underline{\Delta}^n([n])$ whose $d_i$ maps make it look a lot like the $n$-simplex.

**Example 28.10** (An important opposite example)**.** This is just for set-up. Note that there is a functor

$$\Delta \to \mathsf{Spaces}$$

sending $[n]$ to $\Delta^n \subset \mathbb{R}^{n+1}$. This $\Delta^n$ (note it's missing an underline) is the set

$$\{(t_0, t_1, \ldots, t_n) \qquad | \qquad \sum t_i = 1, \qquad t_i \geq 0\}.$$

Any map $f : [n] \to [m]$ induces a continuous map $\Delta^n \to \Delta^m$ by

$$(t_0, \ldots, t_n) \mapsto (s_0, \ldots, s_m), \qquad s_j = \sum_{i \in f^{-1}(j)} t_i.$$

The empty summation equals zero.

**Example 28.11** (The singular complex)**.** Note that any functor $\mathcal{C} \to \mathcal{D}$ is the same information as a functor $\mathcal{C}^{\rm op} \to \mathcal{D}^{\rm op}$ so the above example gives a functor

$$\Delta^{\rm op} \to \mathsf{Spaces}^{\rm op}.$$

Well, any topological space $X$ defines a functor

$$\hom_{\mathsf{Spaces}}(-, X) : \mathsf{Spaces}^{\rm op} \to \mathsf{Sets}$$

where $\hom_{\mathsf{Spaces}}$ is the set of continuous maps. Taking the composite, we find a functor

$$\underline{X} : \Delta^{\rm op} \to \mathsf{Sets}, \qquad [i] \mapsto \hom_{\mathsf{Spaces}}(\Delta^i, X)$$

which we call the *singular complex of* $X$. It's a very important simplicial set. For instance, by taking the free Abelian groups on $\underline{X}_i$ and taking the alternating sum of the maps induced by the $d_i$, one obtains the singular chain complex computing the homology groups of $X$.

**Example 28.12** (Another opposite example)**.** Note that any poset $P$ determines a category: Its objects are elements $p \in P$, and $\hom(p, q)$ is empty when $p \not\leq q$, while $\hom(p, q)$ is a singleton set if $p \leq q$. Composition is forced upon us, and everything works out by the definition of poset.

Then we have a functor

$$\Delta \to \mathsf{Cat}$$

to the category of categories. It sends $[i]$ to the category associated to the poset $[i]$. Any map of posets induces a functor. Details are left to the reader.

**Example 28.13** (The nerve of a category)**.** Let $\mathcal{C}$ be a small category (so its objects form a set, and its morphisms form a set). Define a simplicial set

$$N(\mathcal{C}) : \Delta^{\rm op} \to \mathsf{Sets}$$

by taking

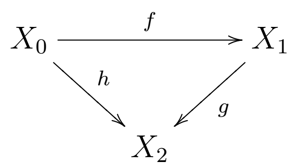
$$[i] \mapsto \mathsf{Fun}([i], \mathcal{C}).$$

This is the set of all functors from $[i]$ to $\mathcal{C}$. By abuse of notation begun above, we are using $[i]$ to denote both a poset and the associated category.

Parsing the definitions, one can see that

(0) $N(\mathcal{C})_0 = N(\mathcal{C})([0])$ is the set of objects of $\mathcal{C}$.
(1) $N(\mathcal{C})_1$ is the set of all morphisms of $\mathcal{C}$. The maps $d_0, d_1 : [0] \to [1]$ induce maps $N(\mathcal{C})_1 \to N(\mathcal{C}_0)$ sending each morphism to its target and source, respectively.

(2) $N(\mathcal{C})_2$ is the set of all commutative triangles




in $\mathcal{C}$. Given such a triangle $T$, the face maps $d_i$ act by

$$d_0T = g \qquad d_1T = h \qquad d_2T = f$$

(3) More generally, $N(\mathcal{C})_n$ is the set of all commutative diagrams in the shape of an $n$-simplex.

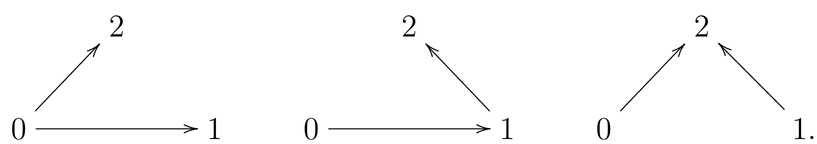
Finally, the degeneracy maps $s_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}$ act by inserting an identity morphism at the $i$th object. In particular, the map $s_0 : N(\mathcal{C})_0 \to N(\mathcal{C})_1$ assigns to each object its identity morphism.

## 2. Categories as simplicial sets

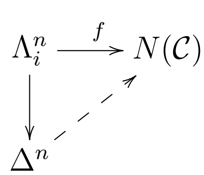
We build on the nerve example from above.

**Definition 28.14.** Let $\Lambda^n_i$ be the simplicial set obtained from $\Delta^n$ by deleting the face opposite the $i$th vertex (and the interior of $\Delta^n$). If $0 < i < n$, $\Lambda^n_i$ is called an *inner horn*.

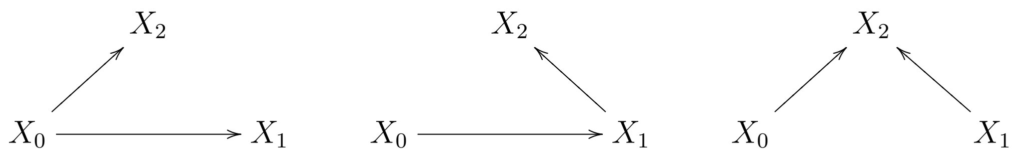
If you like, you can define $\Lambda^n_i$ as the functor $\Delta^{\text{op}} \to \mathsf{Sets}$ obtained by taking the colimit of the functors $\underline{\Delta}^{n-1}$, glued along the functors $\underline{\Delta}^{n-2}$ via the gluing maps suggested by the geometric description above. Here are pictures of the horns $\Lambda^2_0, \Lambda^2_1, \Lambda^2_2$ in that order:




**Proposition 28.15.** Fix $\mathcal{C}$ a small category. Then for any $0 < i < n$, and any map $f : \Lambda^n_i \to N(\mathcal{C})$, there exists a *unique* dotted map making the following diagram commute:



**Remark 28.16.** If you draw this out for $n = 2$, this just means that any pair of morphisms $X_0 \to X_1$ and $X_1 \to X_2$ can be uniquely composed. For $n = 3$, you'll find associativity. More concretely in the $n = 2$ case, what does it mean to be able to find a dotted arrow given a map $f$? Well, if $i = 0, 1, 2$, the map $f$ determines diagrams




respectively. The existence of the dotted arrow means these morphisms can be completed to a commutative triangle. Of course, for $i = 0, 2$, being able to fill any $f$ amounts to finding right and left inverses to any morphism.

**Proposition 28.17.** There is an equivalence of categories between

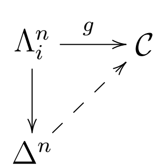
(1) Simplicial sets with unique inner-horn fillings,
(2) small categories.

PROOF. From (2) to (1) is the nerve construction. The opposite direction is given by declaring $X_0$ to be the objects of a category, and morphisms from $x$ to $y$ to be those elements of $X_1$ with $d_1 = x, d_0 = y$. You can define composition via the 1st face of the unique 2-simplex filling $\Lambda^2_1$, and so forth. You can fill in the details. □

## 3. ∞-categories

So we can completely capture categories in terms of simplicial sets. Going all the way back to the opening remarks of this lecture, one should imagine an ∞-category to be something where the inner horn cannot be filled uniquely, but just filled. (So that compositions exist, though they may not be uniquely determined.) That's exactly what we'll do.

**Definition 28.18.** An ∞-category, or weak Kan complex, or quasi-category, is a simplicial set $\mathcal{C}$ such that for any morphism $f : \Lambda^n_i \to \mathcal{C}$ with $0 < i < n$, there exists a dotted arrow making the diagram




commute.

Now, the category of simplicial sets is enriched over itself. Namely, given two simplicial sets $X, Y$, one can define a simplicial set

$$\hom(X, Y)$$

by setting

$$\hom(X, Y)_i := \mathsf{NatTrans}(X \times \underline{\Delta}^i, Y)$$

where the product simplicial set $X \times \underline{\Delta}^i$ is defined using the fact that one can take products of sets. So the righthand side is the *set* of natural transformations.

**Definition 28.19.** Let $\mathcal{C}, \mathcal{D}$ be $\infty$-categories. A functor from $\mathcal{C}$ to $\mathcal{D}$ is just a natural transformation from $\mathcal{C}$ to $\mathcal{D}$. The simplicial set of functors from $\mathcal{C}$ to $\mathcal{D}$ is the simplicial set

$$\hom(\mathcal{C}, \mathcal{D})$$

as defined just now. We will write

$$\mathsf{Fun}(\mathcal{C}, \mathcal{D}) := \hom(\mathcal{C}, \mathcal{D})$$

to emphasize the word "functor."

## 4. Why this model?

I was asked why this model is a good one for $(\infty, 1)$-categories. I will give a few answers:

(1) We have the desirable property (which I haven't proved) that $\mathsf{Fun}(\mathcal{C}, \mathcal{D})$ is again an $\infty$-category. This isn't so obvious to model or prove in other models. Moreover, one can prove that $\mathsf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \simeq \mathsf{Fun}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{E}))$, which is quite easy.

(2) One has easy Grothendieck constructions. A Grothendieck construction is akin to a classifying space construction: A map $f : X \to BG$ is the same thing as a $G$-bundle, so one would like to construct the $G$-bundle out of the data of $f$. Likewise, a functor $F : \mathcal{C} \to \mathsf{Sets}$ of ordinary categories allows one to construct a category $\tilde{\mathcal{C}} \to \mathcal{C}$ whose fiber at $X$ is the set $F(X)$. The $\tilde{\mathcal{C}}$ is called the Grothendieck construction. In our setting, any functor $\mathcal{C} \to \mathsf{Spaces}$ determines a $\tilde{\mathcal{C}} \to \mathcal{C}$. Lurie calls this (un)straightening in Higher Topos Theory.

There is only one thing I dislike about this model. It is the fact that we need degeneracy maps—i.e., we need to say what the $s_i$ do. For instance, we must specify a map

$$s_0 : \mathcal{C}_0 \to \mathcal{C}_1$$

i.e., we must pick out a unit for any object $X \in \mathcal{C}_0$. This is a pain, and somewhat unnatural. For instance, in the Fukaya category, you would never consider “specifying a unit” as part of the data you need to define the Fukaya category.

Another complaint you might have about this model is that it doesn’t generalize easily to define $(\infty, 2)$-categories, but this isn’t such a concern for me. There are plenty of theorems to prove in the $(\infty, 1)$-realm, and we don’t have so many examples of $(\infty, 2)$-categories yet for which we want to develop theories we couldn’t prove using $(\infty, 1)$-language.

Finally, just for reference, some other models for $(\infty, 1)$-categories are:

- Complete Segal spaces (which do have more straightforward generalizations to define $(\infty, n)$-categories for $n \geq 2$),
- Topologically enriched categories
- Simplicially enriched categories
- Simplicial/combinatorial model categories.