Today: More twd local systems

Def: X top sp. local sys on X (w/ val in Chain) is a functor $X \to$ Chain.

Compare: Classically

(old) Def: A local system on X is a locally constant shv. (w/ values in Ab)

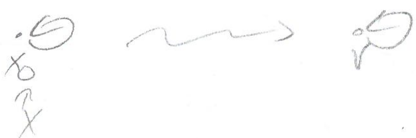
Thm (Riemann-Hilbert) TFAE:

(1) Local sys. w/ val. in Vect_k

(2) Vect. bundles on X w/ flat connection.

(3) Gp hom. $\pi_1(X) \to GL_n$

Compare to (3)



Last time

Def: An ∞-cat is a sset $\mathcal{C}$ s.t. $\forall\, 0<i<n$,

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall f"] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Def$_n$: An ∞-groupoid is a sset $\mathcal{C}$ s.t. $\forall$:

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "\forall f"] \arrow[d] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Why is this a groupoid?

$\Lambda^2_1 \xrightarrow{\beta} \mathcal{C}$, $x_0 \to x_1 \to x_2$ $\rightsquigarrow$

Other horns? $\Lambda^2_0 \to \mathcal{C}$ $\quad x_0 \to x_2,\ x_0 \to x_1 \rightsquigarrow$ filling?

If so $\forall\, x_0 \xrightarrow{h} x_1$, take horn $x_0 \xrightarrow{id} x_0$, $x_0 \xrightarrow{h} x_1$

$\Rightarrow \exists$ filling $\Leftrightarrow h$ admits left inverse!

Λ^2_2 $\quad x_0 \to x_2 \leftarrow x_1 \rightsquigarrow \exists$? filling

Taking $x_1 \xrightarrow{id} x_1 \xleftarrow{h} x_0 \rightsquigarrow \exists$ filling $\Leftrightarrow h$ has right inverse

Thus filling condition for Λ^2_0 & Λ^2_2 is equivalent to $\mathcal{C}$ being a groupoid.

Rk: $\mathcal{C}$ nerve of usual category, being an ∞-gpd is equivalent to $\mathcal{C}$ being a groupoid.

Rk: ∞-groupoid $\Leftrightarrow \forall h \in \mathcal{C}_1$ admits a homotopy inverse.

(∞-gpd are also called Kan complexes)

Ex: X space. Let $\mathrm{Sing}(X) \in sSet$ be sSet sending $[i] \mapsto \mathrm{Map}(\Delta^i, X)$. Then this is an ∞-gpd.

For example $\Lambda^2_0 \to \mathrm{Sing}(X)$ is the data of

$0 \to 2,\ 0 \to 1 \xrightarrow[\text{comp}]{f} X \rightsquigarrow \exists$ filling

Thm (Quillen): $\exists$ an equivalence of homotopy theories

Spaces $\simeq$ Kan complexes

(comp. gen. Hausdorff)

Def: Let $\mathcal{C}$ be an ∞-category, then, $X, Y \in \mathcal{C}_0$

$$\hom(X,Y)_n := \mathrm{Map}^{X,Y}_{sSet}(\Delta^{n+1}, \mathcal{C}) = \left\{ f : \Delta^{n+1} \to \mathcal{C} \text{ s.t. } f(n+1^{st} \text{ vertex}) = Y, \ f((n+1)^{st} \text{ face}) = X \right\}$$

Ex: $n=1$

$$\begin{tikzcd} X \arrow[d, "id_X"'] \arrow[dr, phantom, "\Downarrow H"] \arrow[r, "f"] & Y \\ X \arrow[ur, "g"'] & \end{tikzcd}$$

Lemma: If $\mathcal{C}$ is an ∞-cat., $\hom(X,Y)$ is a Kan complex.

Chain as an ∞-category

There is a construction turning dg-categories into ∞-categories and there is an A_∞-version.

Rk: this gives "correct answer" for Chain_k, k field but not so in general.

(Need to invert quasi-isoms in general)

Def (Lurie): Let $\mathcal{D}$ be a dg-category. The (dg-)nerve of $\mathcal{D}$ is the sset $N(\mathcal{D})$:

$$N(\mathcal{D})_0 = \mathrm{ob}\,\mathcal{D}$$

$$N(\mathcal{D})_1 = \{ f \in \hom^0(X,Y) \text{ some } X, Y \in \mathrm{ob}\,\mathcal{D} \text{ s.t. } df = 0 \}$$

$$N(\mathcal{D})_2 = \left\{ (f_{01}, f_{12}, f_{02}, H) \text{ s.t. } f_{ij} \in N(\mathcal{C})_1, \ dH = f_{02} - f_{12} \circ f_{01} \right\}$$

$$\begin{tikzcd} & X_2 \\ X_0 \arrow[ur, "f_{02}"] \arrow[r, "f_{01}"'] \arrow[r, phantom, shift left=3, "H"] & X_1 \arrow[u, "f_{12}"'] \end{tikzcd}$$

$H \in \hom^{-1}(X_0, X_2)$

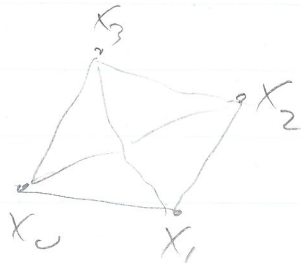
data of a homotopy

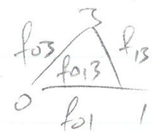
$$N(Q)_n := \begin{cases} \cdot\, f_J & \forall J \subset [n] \quad (f_J \in N(\mathcal{C})_{|J|-1}) \\ \cdot\, H & \end{cases}$$

$$dH = \sum_{i=1}^{n-1} (-1)^{n+1} f_{[n]-\{i\}} + \sum_{j=1}^{n-1} (-1)^j f_{j\ldots n} \circ f_{0\ldots j}$$

$$|f_J| = -(|J|-2)$$

e.g.: $(n=3)$





etc.

$$dH = f_{023} - f_{013} + f$$

RL: A_∞-version:

$$N(Q)_n = \begin{cases} \cdot\, (f_J) & \text{s.t. } f_J \in N(\mathcal{C})_{|J|-1} \\ \cdot\, H & \mu^1 H = \sum (-1)^{i+p} f_{[n]-i} + \end{cases}$$

$$\sum_{d \geq 2} \; \sum_{\substack{\text{contiguous partitions of } [n] \\ [n] = J_1 \cup \cdots \cup J_d}} (-1)^{\Delta} \mu^d (f_{J_d}, \ldots, f_{J_1})$$