9 Nov (Math 277)

<u>Last time.</u> · Every space X gives rise to a Kan cplx, obj points of X (∞-groupoid). hom is paths in X $\forall$ horns

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \mathrm{Sing}(X) \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

· Any dg-cat gives rise to ∞-cat

$$\underset{\text{dg-cat}}{A} \quad \underset{\text{dg-nerve}}{\rightsquigarrow} \quad \underset{\infty\text{-cat}}{N(A)}$$

obj = ob A

hom (X, Y) = closed deg = 0 morphism

2-simplex

$$\left\{ \begin{tikzcd} & x_2 \\ x_0 \arrow[ur, "f_{02}"] \arrow[r, "f_{01}"'] \arrow[ur, phantom, "H", shift left=-3] & x_1 \arrow[u, "f_{12}"'] \end{tikzcd} \right\} = \left\{ \begin{array}{l} H \in \hom^{-1}(x_0, x_2) \\ \text{s.t. } dH = f_{02} - f_{12} f_{01} \end{array} \right\}$$

Def. A local system (of k-chain cplxes) on X is a functor

$$\mathrm{Sing}(X) \longrightarrow \mathrm{Ch}_k$$

Concretely: $\forall x \in X$, a chain cplx V_x,

$\forall\ x \sim y$ a chain map $V_x \to V_y$

$\sim$ a htpy from f_{02} to $f_{12} \circ f_{01}$

$\Delta^2 \to X$

$\longrightarrow$

$\forall\ f: V_{x_0} \to V_{x_1}$ given by local system, γ^{-1} defines $f^{-1}: V_{x_1} \to V_{x_0}$ s.t. $f^{-1} \circ f \overset{\text{htpy}}{\simeq} \mathrm{id}_{V_{x_0}}$.

ξ_0

Q. Is this the same as a complex of sheaves (locally constant)? or sheaf of complexes?

Recall. goal is to accumulate CY-alg of ... using the CFT formalism (category of open string formalism).

outline. Recall $\mathcal{D}$ cat of open strings,

$\hom_{\mathcal{D}}(\mathcal{R}, \sigma) \simeq \{\Sigma\}$

$\mathcal{D}$ was enriched in spaces, so hom

$\forall\, d \in \mathbb{Z}$, we'll construct $\mathcal{D}^d$ = a category enriched in local systems.

$\hom_{\mathcal{D}^d}(\mathcal{R}, \sigma) - (\{\Sigma\}, \det{}^{\otimes d})$.

Once we construct this cat, we apply "cohom/global sections" functor

$$\underset{\substack{\text{enriched}\\ \text{in local}\\ \text{systems}}}{D^d} \xrightarrow{} \underset{\substack{\text{enriched in}\\ \text{cochain cplxes}\\ \text{(i.e., a dg or}\\ A_\infty\text{-cat)}}}{C^*(D^d)}$$

Then we'll define (or see) that

a CY-alg of dim d is a

functor $C^*(D^d) \to \mathrm{Chain}$

Taking $\amalg \to \otimes$.

<u>Things to do:</u>

Ⓐ Tell you what D^d is.

Ⓑ what's "enriched in local systems"

Ⓒ explain $D^d \to C^*(D^d)$

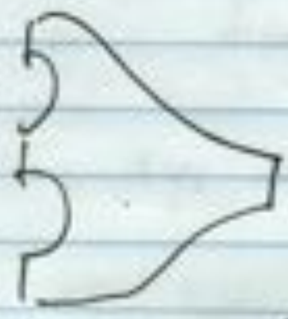
Ⓓ Understand $C^*(D^d) \to \mathrm{Chain}$.

Let's start.

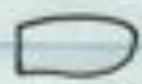
Idea behind (D): roughly, det will assign to each Σ, a cochain cplx concentrated in degree $\pm\big(\chi(\Sigma) - \#(\text{outgoing } I)\big)$.

not sure about sign right now.

Example. (with out



$I \qquad J$ $\longmapsto$ something in deg $= 0$

Example.  $\xmapsto{\det}$ degree ± 1.

$I \qquad \phi$

So $\xmapsto{\det^{\otimes d}}$ degree $\pm d$.

$\hom(D^d) = (\{\Sigma\}, \det^{\otimes d})$

This is the degree d map

$$A \to k[\pm d].$$

Upshot. mult maps will all be deg $= 0$ (still), but trace maps will have deg $= 2d$.

To do (c),

Prop. $\exists$ a functor

$$\text{Loc Sys} \to \text{Chain}$$

called "chains" w/ local coeff. It is given by

$$(X \xrightarrow{E} \text{Chain}) \mapsto \operatorname{colim}_X E$$

Intuition. If we have $(X \xrightarrow{\text{const}} \text{Chain}) \mapsto N_*(X;k)$ (probably).

This functor is symmetric monoidal.

We need to define: cat of local systems and colimit.

<u>Defn</u>. The ∞-cat of local systems is the ∞-cat of spaces$/$Chain.

∞-cat

This is a general construction: whenever we have a functor

$$\begin{array}{ccc} S & \xrightarrow{i} & D \\ & & \cup \\ & & C \end{array}$$

$\exists$ new ∞-cat called $S/_C$ (the "comma" category or "slice," "over" ...)

In our setting S = Spaces, D = ∞-cat, $C \in D$ called ∞-Chain.

$i: X \mapsto \mathrm{Sing}(X)$.

An object of S/D is a pair

$$(S, \ j : i(S) \to C)$$

For our setting

$$(X, \ \underbrace{f : X \to \text{Chain}}_{\text{a local system}}).$$

Rmk. Can phrase as fiber product

A morphism from

$$(S_1, j_1) \text{ to } (S_2, j_2)$$

is a triangle

$$\begin{tikzcd} i(S_1) \arrow[rr, "i(f)"] \arrow[dr, "j_1"'] & & i(S_2) \arrow[dl, "j_2"] \\ & C & \end{tikzcd}$$

in D. A k-simplex in S/C is a choice of k-simplex in S, and a map

$\Delta^{k+1} \to \mathcal{C}$ in the obvious shape

In our example, a morphism from (X, ϵ_X) to (Y, ϵ_Y) is a diagram of ∞-cat

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\epsilon_X"'] & \arrow[d, phantom, "\Rightarrow"] & Y \arrow[dl, "\epsilon_Y"] \\
& \text{chain} &
\end{tikzcd}$$

i.e., a htpy from ϵ_X to $f^*\epsilon_Y$.

local system have a sym $\otimes$ str induced by $\times$ of spaces $\otimes$ of chains.

Given (X,ϵ_X), (Y,ϵ_Y). That is —

Consider

$$X \times Y \xrightarrow{\epsilon_X \times \epsilon_Y} \begin{matrix} \text{Chain} \\ \times \\ \text{Chain} \end{matrix} \xrightarrow{\otimes} \text{Chain}$$

Concretely,

$$(x,y) \longmapsto \epsilon_X(x) \otimes \epsilon_Y(y)$$

When $X = Y = pt$ this is the usual tensor product of chain cplxes.

Colimits. Given a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ of ∞-cat, a colim of F is an initial object in the cat of objects

Rmk. If X connected, a funct $X \to \text{Chan}$ is the same as a module over ΩX.

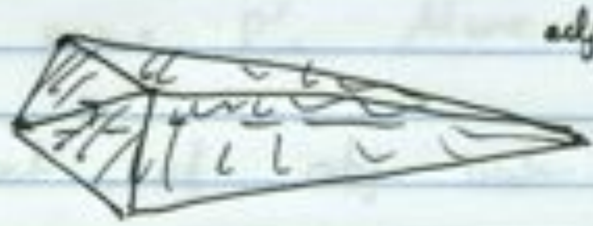
$$\begin{matrix} \Omega X\text{-Mod} \\ \wr\!\uparrow \\ \text{Fun}(X, \text{Chain}) \end{matrix} \qquad \text{equiv of cat}$$

living under F.

<u>Def.</u> (regardless of whether $\mathcal{C}, D$ are ∞-cat or not). Given an ∞-cat $\mathcal{C}$, we can construct a new one, called

$$\mathcal{C}^{\triangleright} = \mathcal{C} \underset{\substack{\uparrow \\ \text{adjoin}}}{\star} pt.$$

<u>Picture.</u>



$\mathcal{C}$ $\qquad\qquad$ pt

A ~~functor~~ functor $\tilde{F}$ with choice of H

$$\begin{tikzcd} \mathcal{C}^{\triangleright} \arrow[r, "\tilde{F}"] & D \\ \mathcal{C} \arrow[u, hook, "\text{inc. to base}"'] \arrow[ur, bend right, "F"'] \arrow[ur, phantom, "H"] & \end{tikzcd}$$

is called an <u>object under F.</u>

<u>Example.</u> If $\mathcal{C} = pt$,

$$\mathcal{C}^{\triangleright} = \bullet \longrightarrow \bullet$$

and given

$$F : \mathcal{C} \longrightarrow D$$

$$pt \longrightarrow D$$

an obj under F is a morphism $D \to D'$ for some D'. More generally, the ∞-cat of obj under F has n-simplices

$$\begin{tikzcd}
\mathcal{C} \star \Delta^n \arrow[r, "\widetilde{F}"] & D \\
\mathcal{C} \arrow[u, hook, "H"'] \arrow[ur, "F"'] &
\end{tikzcd}$$

<u>Defn.</u> An initial object of $\mathcal{E}$ is an object $X \in \mathcal{E}_0$ s.t. $\hom(X, Y)$ is contractible $\forall Y$.