11/18.

Recall.

Thm (Costello)

(3) Given a d-dim'l CY cat

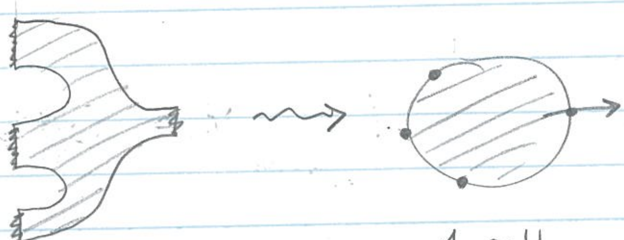
$$\begin{tikzcd} \mathcal{O}^d \arrow[r, "\mathcal{C}"] \arrow[d] & \text{Chain} \\ \mathcal{OC}^d \arrow[ur, dashed, "LKE(\mathcal{C})"'] & \end{tikzcd}$$

$$LKE(\mathcal{C})(S^1) = CH_*(\mathcal{C})$$

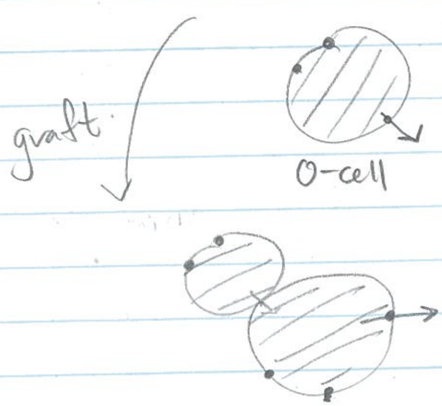
Today. Sketch of proof.

Recall. We've replaced $\begin{matrix}\mathcal{O} \\ \mathcal{OC}\end{matrix} \rightsquigarrow \begin{matrix}\mathcal{D}_{open} \\ \mathcal{D}\end{matrix}$

by cellular models



1-cell



1-cell.

As dg-categories, $C_* \mathcal{O} \simeq \mathcal{D}_{open}$,
$C_*(\mathcal{OC}) \simeq \mathcal{D}$.

Recall $\quad LKE(\Phi)(S') := \hom_{\mathcal{OC}}(-,S') \overset{\mathbb{L}}{\underset{\mathcal{O}^d}{\otimes}} \Phi(-)$

$\mathcal{O} \xrightarrow{C=\Phi} \text{Chain}$

What do we mean by $\underset{\mathcal{O}_d}{\otimes}$?

(1) Naive/wrong definition.
Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are dg-cats
and we have functors $M : \mathcal{A} \otimes \mathcal{B}^{op} \to \text{Chain}$
$N : \mathcal{B} \otimes \mathcal{C}^{op} \to \text{Chain}$.

$A \otimes B$ is a category where $ob(A \otimes B) = ob A \times ob B$
$\hom((a_0,b_0),(a_1,b_1)) := \hom_A(a_0,a_1) \otimes \hom_B(b_0,b_1)$.

Then $M \underset{B}{\overset{naive}{\otimes}} N : \mathcal{A} \otimes \mathcal{C}^{op} \to \text{Chain}$

$$(A,C) \mapsto \operatorname{coeq}\left(\bigoplus_{B_1,B_2 \in ob B} M(A,B_2) \otimes \hom_B(B_1,B_2) \otimes N(B_1,C) \rightrightarrows \bigoplus_B M(A,B) \otimes N(B,C) \right)$$

Ex. If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ have only 1 object, this recovers the usual notion of $\underset{B}{\otimes}$.

NOT say what the correct $\overset{\mathbb{L}}{\underset{B}{\otimes}}$ is, but claim:

If M is a bimodule s.t. $N \simeq N'$ quasiequivalent

"M is flat" $\left\{ \Rightarrow M \underset{B}{\otimes} N \xrightarrow{\sim} M \underset{B}{\otimes} N' \right.$, then

$$M \overset{\text{naive}}{\underset{B}{\otimes}} - \simeq M \overset{\mathbb{L}}{\underset{B}{\otimes}} - .$$

Outline of proof of (3).

Note: since $\exists$ functor $\mathcal{O} \overset{j}{\hookrightarrow} \mathcal{OC}$, have

$\mathcal{OC} := \hom_{\mathcal{OC}} : \mathcal{OC} \otimes \mathcal{O}^{op} \longrightarrow \text{Chain}$

abuse notation!

$$(X, Y) \longmapsto \hom_{\mathcal{OC}}(j(Y), X)$$

so $\mathcal{OC}$ is a $(\mathcal{OC}, \mathcal{O})$-bimodule.

and $LKE(\Phi) = {}_{\mathcal{OC}}\mathcal{OC} \underset{\mathcal{O}}{\otimes} \Phi$.

Now,

$$\mathcal{OC} \overset{\mathbb{L}}{\underset{\mathcal{O}}{\otimes}} \Phi \simeq \mathcal{D} \overset{\mathbb{L}}{\underset{\mathcal{D}_{open}}{\otimes}} \Phi$$

Thm.^{A} $\mathcal{D}$ is flat / $\mathcal{D}_{open}$

(explicit quasiequivalence involving generators, relations, filtration, associated graded, ⋮

So $\mathcal{OC} \overset{\mathbb{L}}{\underset{\mathcal{O}}{\otimes}} \Phi \simeq \mathcal{D} \overset{\text{naive}}{\underset{\mathcal{D}_{open}}{\otimes}} \Phi$.

Thm.^{B} $\underset{\substack{| \\ \text{cats over} \\ \text{gr Vect}}}{\mathcal{D}} \simeq \text{Free}_{\text{ob}\,\mathcal{OC}-\mathcal{D}_{open}} \langle \text{Annuli} \rangle / \sim$ ← relations only involve annuli and morphisms from $\mathcal{D}^+$.

Recall:

$\mathcal{O}$

S^1

$\mathcal{D}_{open} \underset{full}{\subset} \mathcal{D} \simeq C_* OC.$

only opens

$\cup$ not full

$\mathcal{D}^+$

homs are $\sqcup$ of disks w/ unique outpoint.

<u>Define</u>: $\mathcal{D}_{sub} := \mathrm{Free}_{ob\, OC - \mathcal{D}^+}\langle \text{Annuli} \rangle /\sim$ same relations as in thm.

Compare: If $A^+ \overset{\text{subalgebra}}{\subset} A_{open}$, and $M \cong \mathrm{Free}\langle A \rangle / R$

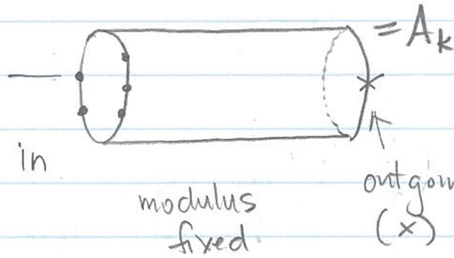
then $M_{sub} := \mathrm{Free}_{A^+}\langle A \rangle / R$ R modules A and elts of A^+.

<u>and</u> $M \cong M_{sub} \overset{\text{naive}}{\underset{A^+}{\otimes}} A_{open}$.

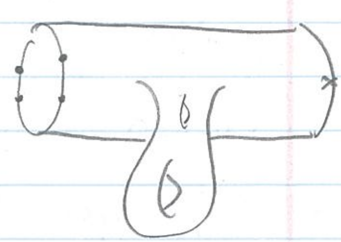
$$\text{So } \mathcal{D} \overset{\text{naive}}{\underset{\mathcal{D}_{open}}{\otimes}} \Phi \simeq (\mathcal{D}_{sub} \underset{\mathcal{D}^+}{\otimes} \mathcal{D}_{open}) \underset{\mathcal{D}_{open}}{\otimes} \Phi$$

$$\simeq \mathcal{D}_{sub} \underset{\mathcal{D}^+}{\otimes} \Phi$$

Let's explain Theorem B and compute. In our cellular model for OC, we have chains of the form.

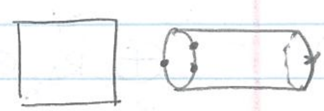


and



Claim $\to$ ||

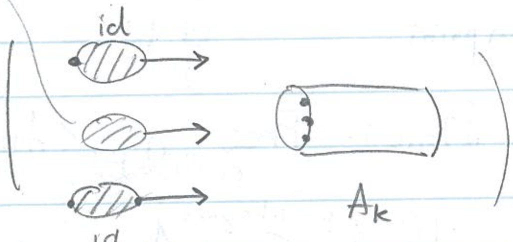
morphisms from $\mathcal{D}_{open}$



More explicitly,

$$\hom_{\mathcal{D}}(B, S') = \bigoplus_{B=B_0, B_1, \dots, B_n} k_{A_k} \otimes \hom_{\mathcal{D}_{open}}(B_n, B_{n-1}) \otimes \cdots \otimes \hom(B, B_1) \Big/ \sim$$

no incoming ith spot with relation



$$= \begin{cases} 0 & \text{if } i \neq 0 \\ A_k. & \end{cases}$$

and relations from the category.

Computation:

$$\mathcal{D}_{sub} \otimes_{\mathcal{D}_+} \Phi(S') = \mathrm{coeq}\left(\bigoplus_{B_0, B_1} \hom_{\mathcal{D}_{sub}}(B_1, S') \otimes \hom_{\mathcal{D}}(B_0, B_1) \otimes \Phi(B_0) \rightrightarrows \bigoplus_{B} \hom(B, S') \otimes \Phi(B) \right)$$

$$\overbrace{\underset{\text{annulus}}{A_k} \circ \underbrace{f_1 \circ \cdots \circ f_n}_{\text{homs in } \mathcal{D}^+}}^{\hom(B,S')} \otimes \overset{\Phi(B)}{g}$$

$$\mapsto$$

$$A_k \otimes f_n^* \cdots f_1^* g.$$

So we have an injection

degree $\Sigma(k-1)$

$$\bigoplus_{B} k_{A_k} \otimes \Phi(B) \Big/ \text{rel'n} \longrightarrow \mathrm{coeq}.$$

$B \in \mathrm{ob}\, \mathcal{D}^+$

$B = \mathbb{R}^{\sqcup k} \Rightarrow \Phi(B) = \Phi(\mathbb{R})^{\otimes k}$

$=: A^{\otimes k}$

so $\mathrm{LHS} = \bigoplus_k A^{\otimes k}\,[\text{linear in } k] / \text{rel'n.}$

$f_0 \otimes \cdots \otimes f_k = 0$
if $f_i = id_A$, $i > 0$.

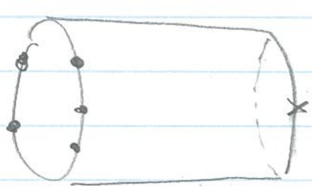
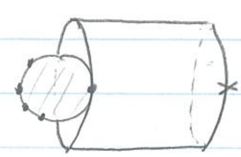
= normalized Hochschild complex
(as graded v.s.)

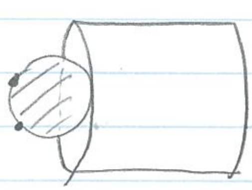
In fact, $\bigoplus_B k_{A_k} \otimes \Phi(B) \longrightarrow \mathrm{coeq}$

is surjective too (e.g. by taking $f_0 = id$).

$\therefore$ iso.

Claim The differentials in the cellular chain complex are precisely the differentials in the Hochschild complex.

d () = $\sum$ 

the only thing that survives. 

$\shortparallel$

differential in Hochschild cplx

Ref. Costello's "Top Conf F. Th & C-Y Cats"

Why no higher mults?

Yoneda embedding: $\mathcal{C} \longrightarrow \mathrm{Fun}_{A_\infty}(\mathcal{C}^{op}, \mathrm{Chain})$
full & faithful.

$\Rightarrow$ A_∞-cat are equivalent to some dg-cat.

Chain has no higher mult.