Kevin Sackel — Fukaya Categories — 20 Nov 2015

- Meta-note: Yau-Calabi structures will be covered in Piazza

- Today: Different kinds of Fuk

  Focus: Fukaya-Seidel category of Lefschetz fibrations

- Recall: Kontsevich's HMS conjecture says:

  $\forall X$ with J, ω (some restrictions)

  $\exists X^\vee$ with $\omega^\vee, J^\vee$

  s.t.

$$D^\pi \mathrm{Fuk}(X, \omega) \simeq D^b \mathrm{Coh}(X^\vee, J^\vee)$$

and

$$D^b \mathrm{Coh}(X, J) \simeq D^\pi \mathrm{Fuk}(X^\vee, \omega^\vee)$$

- Example: $X = (\mathbb{CP}^1, \omega, J)$

  Rmk: Not CY, but is Fano. Can HMS be proved for Fano? Toric Fano? ↳ yes

  Don't know how to handle $D^\pi \mathrm{Fuk}(\mathbb{CP}^1)$ because we didn't talk about $\omega \neq d\Theta$

  But $D^b \mathrm{Coh}(\mathbb{CP}^1)$ is very well understood.

  Mirror? $X^\vee = (\mathbb{C}^\times, \overset{d\Theta}{\ddot{\omega}}, J, W \in \mathcal{O}(X^\vee))$

  where

$$W : \mathbb{C}^\times \longrightarrow \mathbb{C}$$
$$z \longmapsto z + \frac{1}{z}$$

  We will motivate this below

  $D^\pi MF(X^\vee, W)$ "matrix factor" $FS(X^\vee, W)$ "Fukaya-Seidel category"

- Fact

  (o) Let $S^1 \subset \mathbb{CP}^1$. Note that S^1 can be displaced from itself by an area-preserving automorphism of $\mathbb{CP}^1$ unless S^1 divides $\mathbb{CP}^1$ into 2 equal-area halves (*)

  So whatever $\mathrm{Fuk}(\mathbb{CP}^1)$ is, any S^1 not doing (*) should be $\simeq 0$.

· Facts (ctd)

(1) Fix S^1 = equator. Consider the moduli space of flat $\mathcal{U}(1)$ bdles on S^1

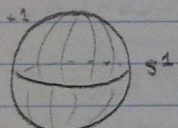
i.e. $\hom(\pi_1 S^1, \mathcal{U}(1)) \cong \mathcal{U}(1) \subset \mathbb{C}^\times$

(2) Suitably defined, the moduli space

$$\{(\text{Special circles}, \nabla^{flat})\}/\sim \;\cong\; \mathbb{C}^\times$$

Ham. isotopy; equiv of flat $\mathcal{U}(1)$-bundles.

And for each point in this moduli space, can compute the monodromies of holo^c discs in S^1



$$\exp^{\text{monodromy}} = z + \frac{1}{z}$$

where $z \in \mathbb{C}^\times \cong$ moduli space

$$u : (D^2, \partial D^2) \longrightarrow (\overset{S^2}{X}, \overset{S^1}{L})$$

$\rightsquigarrow$ Area(u) = elt of $\mathcal{U}(1)$ determined by ∇

· Rmk: If you try to define Fuk$(\mathbb{CP}^1)$, you'll run into a problem with grading ...

In general, $CF(L_0, L_1)$ is $\mathbb{Z}/2\mathbb{Z}$-graded.

· Note: The same structure appears in $MF(X^\vee, W)$

By definition,

$$\text{ob}\, MF(X^\vee, W) = \left\{ d^1 \circlearrowleft \begin{matrix} M^1 \\ M^0 \end{matrix} \circlearrowright d^0 \;\; \begin{matrix} \text{where } M^0, M^1 \text{ are projective modules} \\ \text{over } \mathcal{O}_{X^\vee} \text{ and } d^0 d^1 = d^1 d^0 = \underset{\text{matrix}}{W} \end{matrix} \right\}$$

and

$$\hom(M^\cdot, N^\cdot) = \{f : M^\cdot \to N^\cdot\} \oplus \{f : M^\cdot \to N^{\cdot+1}\}$$

· Today: FS category for Lefschetz fibrations

· Defn: A Lefschetz fibration is the data of

$$E \xrightarrow{\pi} \mathbb{C}$$

with E complex, π holomorphic, and π generic in the sense that π is $\mathbb{C}$-Morse ($\pi = \sum z_i^2$ in nbhd of $x \in \text{Crit}\,\pi$)

Fukaya Cats

20 Nov 2015

- <u>Rmk</u>: More generally, could replace base $\mathbb{C}$ w/ and Riemann surface with ∂.

<u>Rmk</u>: LFs have some symplectic structure on fibers

- <u>Example</u>: $\mathbb{C}^\times \xrightarrow{z+\frac{1}{z}} \mathbb{C}$

⊛ Fix this! → If E comes with J-compatible ω, then get metric g

- <u>Note</u>: Lefschetz thimbles

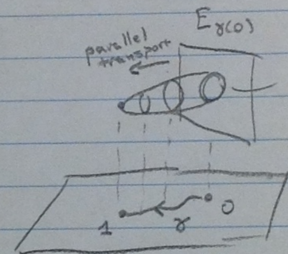
  Fix an embedding $\gamma : [0,1] \longrightarrow \mathbb{C}$

  s.t. $\gamma(1)$ is a critical value of π

  For simplicity, assume $\pi : \mathrm{Crit}(\pi) \xrightarrow{?} \mathrm{Critval}(\pi)$ is bijection.

<u>Lemma</u>: $\exists!$ Lagrangian disk Δ_γ in E s.t. Δ_γ is parallel transport of Lagrangian sphere in $E_{\gamma(0)}$.

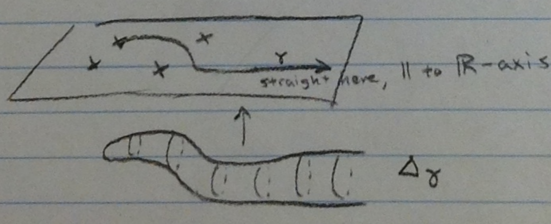
<u>Defn</u>: this is called $V_{\gamma(0)}$, the <u>vanishing cycle</u>



- Defn/ <u>Note</u>: FS category will have two kinds of objects

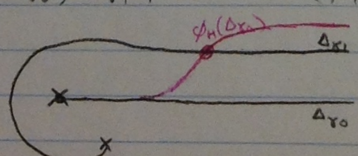
  (1) Compact exact Lagrangians in E

  (2) Thimbles for curves $\gamma : (-\infty, 1] \to \mathbb{C}$ s.t. γ eventually <u>actually</u> parallel to real axis



<u>Homs?</u>

$$\hom(\Delta_{\gamma_0}, \Delta_{\gamma_1}) = CF^*(\phi_H(\Delta_{\gamma_0}), \Delta_{\gamma_1}) \cong CF^*_{E_z}(V_0, V_1)$$



→ H linear outside compact set K

· **Q**: Why $\hom(\Delta_{\gamma_0}, \Delta_{\gamma_1}) \overset{?}{=} CF^*_{E_z}(V_0, V_1)$

Obviously same generators!

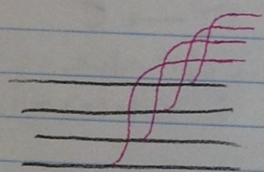
To compute μ

$$\begin{tikzcd} u: \mathbb{R}\times[0,1] \arrow[r, "J\text{-holo}^c"] \arrow[rr, bend right] & E \arrow[r, "\pi"] & \mathbb{C} \end{tikzcd}$$

this has to be constant!! (max principle...) w/ ∂ condition

· A_∞-structure? To compute μ^k:

to CVs $\leftarrow$



Compute μ^k for these perturbed $\phi_{H_i}(\Delta_{K_i})$

Claim: $\mu^k = \mu^k_{E_z}$ — some fiber