11/23

Today: More about the Fukaya-Seidel cat.

Recall. Fix a symplectic mfld $(E, \omega = d\theta)$
and a Lefschetz fibration $E \xrightarrow{\pi} \mathbb{C}$.

$\dim_{\mathbb{C}} E = n+1$.

Here, E has a almost cplx structure J, compatible with ω, and π is J-holomorphic
$$D\pi \circ J = J_{\mathbb{C}} \circ D\pi.$$

The Lefschetz condition says that
- Near every $x \in \mathrm{Crit}(\pi)$, J is integrable.
- $\exists$ holom coords about $x \in \mathrm{Crit}(\pi)$ s.t.
$$\pi = \sum_{i=1}^{n+1} x_i^2 \qquad \text{(complex analogue of "Morse")}$$

Rmk. $D\pi \circ J = J_{\mathbb{C}} \circ D\pi \implies \forall$ regular value $y \in \mathbb{C}$, $E_y := \pi^{-1}(y)$ is also an exact symplectic mfld with $\omega_{E_y} := \omega|_{E_y}$.

Idea (Kontsevich, Seidel)
Study version of $\mathcal{F}uk(E)$ via $\mathcal{F}uk(E_y)$.

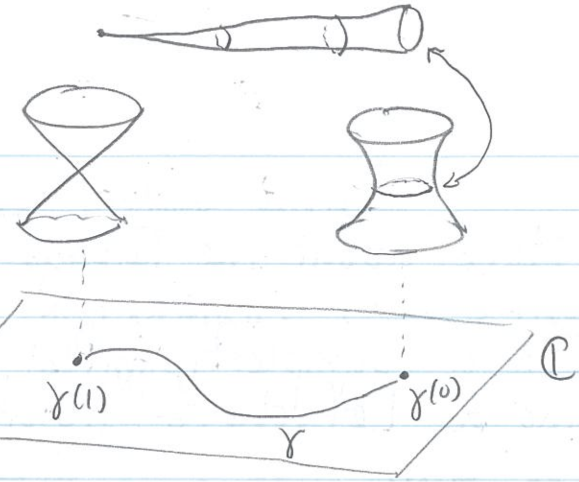
(all fibres are symplectomorphic)

Recall. Fix $\gamma : \cancel{[0,1]}\ (-\infty, 1] \longrightarrow \mathbb{C}$ smth embedding
s.t. $\gamma(1) \in \mathrm{CritVal}(\pi)$, $\gamma(t)$ reg. for $t \neq 1$.

(For simplicity, assume $\mathrm{Crit}(\pi) \xrightarrow{\cong} \mathrm{CritVal}(\pi)$.)

Then, $\exists !$ Lagrangian disk $\Delta_\gamma \overset{\text{diff.}}{\cong} D^{n+1} \subset E$ s.t.
Δ_γ is obtained via parallel transport of a Lagrangian sphere along γ.

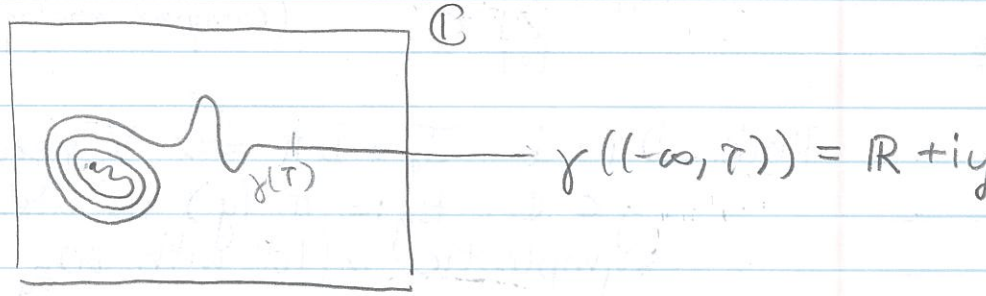
E:



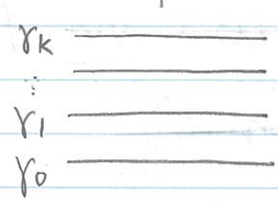
The Fukaya-Seidel category of (E,π)

has objects

- compact exact Lagrangians $L \subset E$
- Δ_γ for γ parallel to $\mathbb{R}$ outside a cpt set, with $\operatorname{Re}\gamma \geq$ some C.

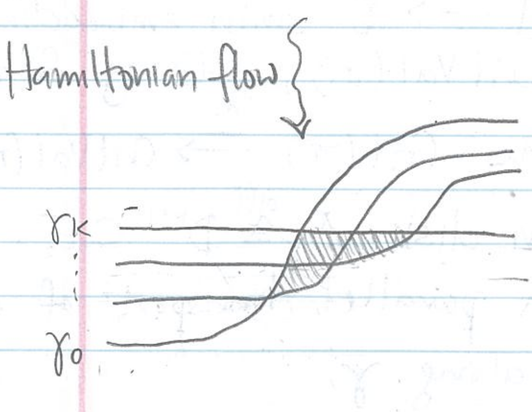


To compute μ^k operations for $\Delta_{\gamma_0}, \dots, \Delta_{\gamma_k}$,

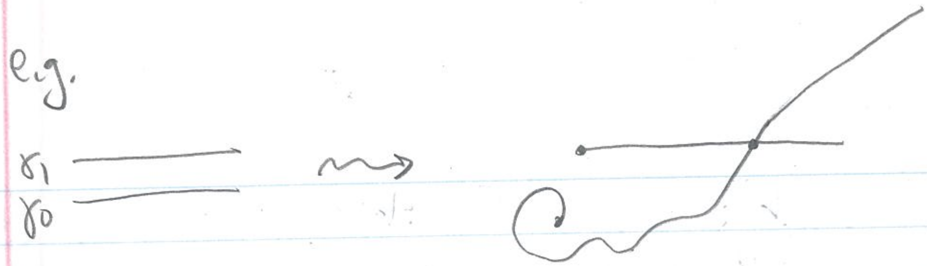


Apply such a fiber preserving Ham. flow, and compute μ^k_E for

$$\phi_{H_k}(\Delta_{\gamma_k})$$
$$\vdots$$
$$\phi_{H_0}(\Delta_{\gamma_0})$$



e.g.



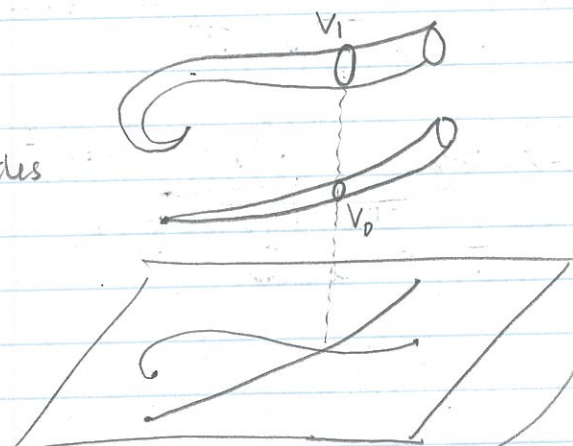
where $\Delta_{\gamma_1} \cap \phi_{H_0}(\Delta_{\gamma_0}) \subset E_y$

Let $V_1 := \Delta_{\gamma_1} \cap E_y$

$V_0 := \Delta_{\gamma_0} \cap E_y$.

thimbles

vanishing cycles

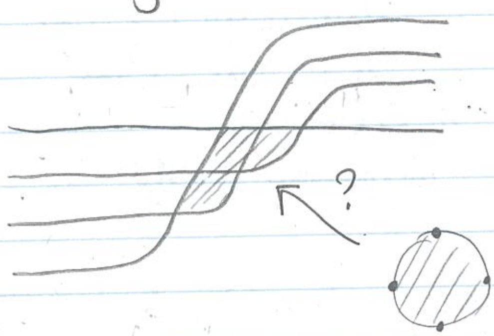


then,

$CF_{E_y}(V_0, V_1) \simeq CF_E(\phi_{H_0}(\Delta_{\gamma_0}), \Delta_{\gamma_1})$

Further, $\mu^k_E = \mu^k$, i.e. μ^k of $\Delta_{\gamma_i} \subset E$ is μ^k of $V_i \subset E_y$.

Why?

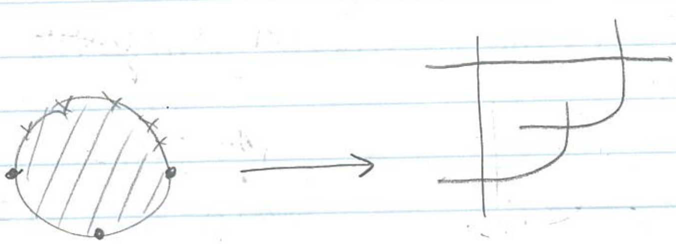


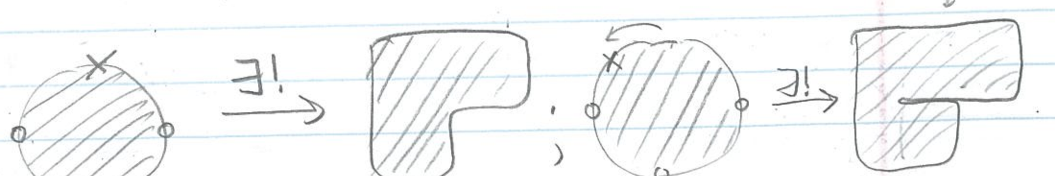
Ex: μ^3.

Claim: $\{$hol. maps $S =$ with fixed $\mathbb{C}$-structure$\}$

$\|$

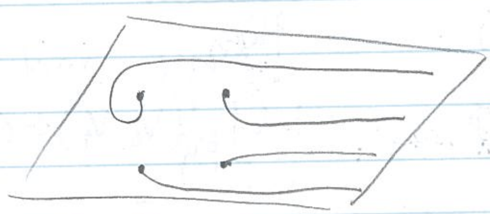
$*$





How about $L \subset E$ that are compact?
(in particular, $L \neq \Delta_\gamma$).

Thm (Seidel)
Fix a collection $\gamma_1, \dots, \gamma_k$, $k = |\mathrm{CritVal}|$
($\gamma_i(1) = p_i$, $p_i \in \mathrm{CritVal}$, for $i = 1, \dots, k$).

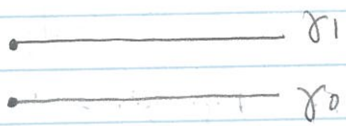


Then,
$$D^\pi\langle \Delta_{\gamma_1}, \dots, \Delta_{\gamma_k} \rangle \simeq D^\pi FS(E, \pi)$$

i.e. (Floer theory of Lefschetz thimbles + monodromy recovers FS cat.)

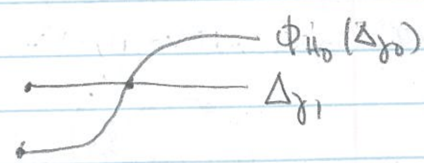
gives self-symplectomorphism of fiber
determines Lagrangian sphere
apply Dehn twist.

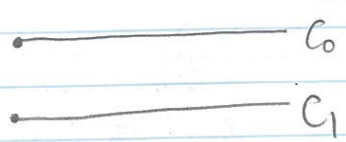
algebraically: generates cones to recover objects in FS.



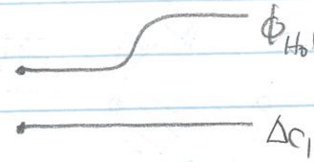
flow

$$\hom(\Delta_{\gamma_0}, \Delta_{\gamma_1}) := CF_E(\phi_{H_0}(\Delta_{\gamma_0}), \Delta_{\gamma_1})$$





$$\hom(\Delta_{c_0}, \Delta_{c_1}) := CF_E(\phi_{H_0}(\Delta_{c_0}), \Delta_{c_1})$$



$\rightsquigarrow$ no homs!

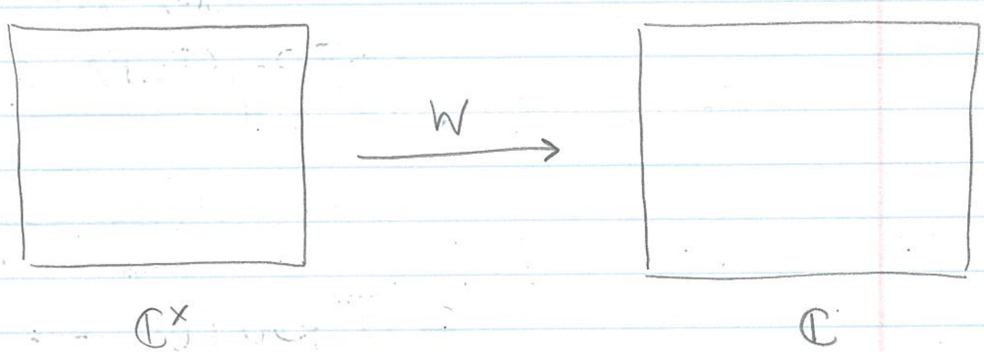
This is an example of a directed category analogue of "exceptional collections"

Rmk. $\mathrm{End}(\Delta_\gamma) \simeq k$.

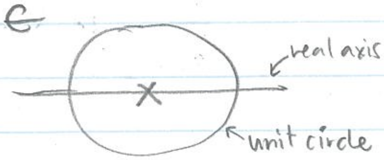
<u>This summer</u> (Witten, Seidel, ...)
Complex Morse theory arises as a <u>physical system</u>.
$\parallel\!\!\mid$
Picard-Lefschetz theory

<u>Thm</u> $D^b\mathrm{Coh}(\mathbb{P}^1) \simeq FS(\mathbb{C}^\times, \pi = W : z \mapsto z + \frac{1}{z})$

Pf. $\mathrm{Crit}(W) = \{z = \pm 1\}$.
$\mathrm{CritVal}(W) = \{\pm 2\}$

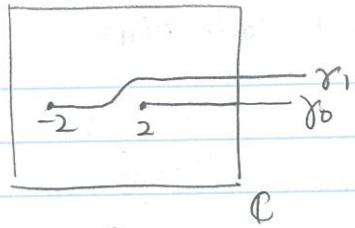


When is $W(z) \in \mathbb{R}$?
A: when $z \in$



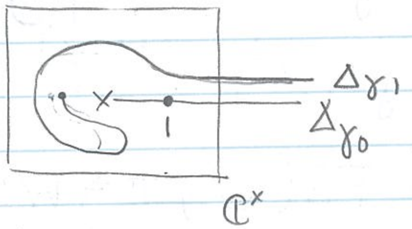
b/c $W(x+iy) = (\text{real}) + iy\left(1 - \frac{1}{x^2+y^2}\right)$.

Choose



$\pi^{-1}\Big\downarrow$

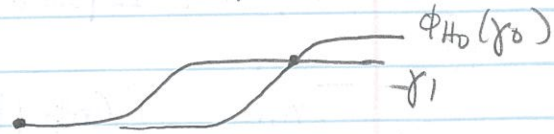
might be wrong



Claim:

$$CF^*(\phi_{H_0}(\Delta_{\gamma_0}), \Delta_{\gamma_1}) \cong k \oplus k \text{ deg } 0$$

w/ no differential.



So by Seidel's thm,

$$D^\pi FS(\mathbb{C}^\times, W) \simeq D^\pi \langle \Delta_{\gamma_0}, \Delta_{\gamma_1} \rangle$$

$$\simeq \underset{(A_\infty^-)}{D^\pi Rep}(\underbrace{End(\Delta_{\gamma_0} \oplus \Delta_{\gamma_1})}_{\shortparallel})$$

$$\text{id} \circlearrowright \underset{\Delta_{\gamma_0}}{\bullet} \rightrightarrows \underset{\Delta_{\gamma_1}}{\bullet} \circlearrowleft \text{id}$$

$$\simeq D^\pi Rep(\underset{\text{quiver}}{Q} = \bullet \rightrightarrows \bullet)$$

$$\underset{(\text{Beilinson})}{\overset{\text{Thm}}{\simeq}} D^b Coh(\mathbb{P}^1)$$

$Ext(\mathcal{O} \oplus \mathcal{O}(1))$

$\square$

This is one half of mirror symmetry for $\mathbb{P}^1$.

exception: Lagrangian forms.

These proofs are unsatisfying because it's <u>not geometric</u>; always a highly nontrivial algebraic step that makes it look like an accident.

Next time: $FS \overset{??}{=}$ gluing of $\underbrace{(\text{compact part})}_{\text{algebraic description}}$ & (noncompact part)