11/30.

Nadler-Zaslow:

$$\mathrm{Const}(X) \cong \text{"Fuk"}(T^*X)$$

constructible derived cat

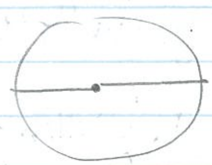
some mfld

equipped w/ standard symplectic structure.

$\mathrm{Const}(X) \subset$ bdd. derived cat of cplxes of sheaves of $\mathbb{C}$-vector spaces.

cat of cplxes w/ constructible cohomology.

i.e. $C^\bullet \subset \mathrm{Const}(X)$ if $\mathcal{H}^i(C^\bullet)$ constructible



$C^\bullet$ is a local system restricted to each stratum of X.

Upshot: have six functors formalism

$X \xrightarrow{f} Y \rightsquigarrow f_*, f^*, f_!, f^!, \ldots$

all derived functors.

e.g. $X \xrightarrow{f} *$

$f_* \mathbb{C}$ is the cochain cplx of X.

In general, f_* is a relative version of this

$f_!$ computes cohomology w/ cpt support — relative cohomology of $(\bar{X}, \partial \bar{X})$ for some compactification $\bar{X}$

- $f_*, f_!$ agree for proper morphisms
- $f^*, f^!$ agree on open subsets.
- $f^* \dashv f_*$, $f_! \dashv f^!$.

$$U \underset{\text{open}}{\overset{j}{\subset}} X \overset{i}{\hookleftarrow} Z = X \setminus U$$

$$i_* i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow j_* j^* \mathcal{F}$$ some exact sequence

section supported on Z $\longrightarrow$ sections $\longrightarrow$ sections on U.

$\Rightarrow$ Const(X) $\underbrace{\text{generated}}$ by $j_* \mathbb{C}_U$.

every object can be written as a successive cones, shifts.

(exact sequence reduces to
$i : M \longrightarrow X$, M any submfld.

By taking a finer stratification, can take $\mathcal{L} = \mathbb{C}$.
can reduce to open case

$i_* \mathbb{C}_U$

$$\begin{tikzcd} & \mathrm{Fuk}(T^*M) \\ \mathrm{Open}(X) \arrow[r] \arrow[ur] & \mathrm{Const}(X) \arrow[u, dashed] \end{tikzcd}$$

To give map Const$(X) \to$ Fuk(T^*M),
enough to define it on Open(X).

Take for $\mathbb{C}_X \rightsquigarrow$ zero section.

$$\mathrm{Hom}(\mathbb{C}_X, \mathbb{C}_X) \cong H^\bullet(X)$$

$$\mathrm{Hom}(\text{zero}, \text{zero}) \cong H^\bullet(X) \quad \text{by PSS.}$$

This is a categorification of the singular support map.

We have a functor

$$SS : K^0(\mathrm{Const}) \longrightarrow \mathbb{Z}(\text{closed submflds of } X)$$

assigns to every constructible sheaf a conical Lagrangian cycle on T^*X that is a finite lin. comb. of conormal bundles $T^*_Y X$.

e.g. $SS(\mathbb{C}_X) =$ zero section.

Z closed, $SS(i_*\mathbb{C}_Z) = T^*_Z X$

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SS measures "singular co-directions"

U open, f a defining function for $X \setminus U$ ($f = 0$ exactly on $X \setminus U$)

$$SS(i_*\mathbb{C}_U) = \lim_{t \to 0} L_{d\log f} \qquad d\log f \subset T^*X.$$

$L_{d\log f}$ are Hamiltonian isotopic.

Open (X)

objects: (U, f) U open w/ defining ftn f.

our functor

$$\mathrm{Open}(X) \longrightarrow \mathrm{Fuk}(T^*X)$$

sends (U, f) to $d\log f \subset T^*X$

(U, f): iso for different f

$d\log f$: Ham. isotopic for different f.

Morphisms in Open(X).

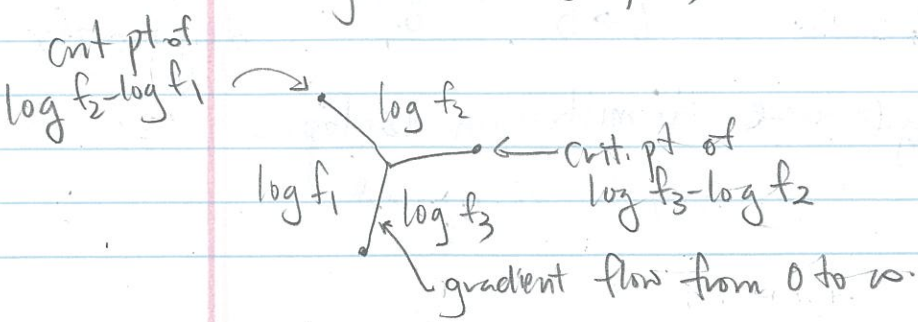
$$\mathrm{Hom}(i_{1*}\mathbb{C}_{U_1}, i_{2*}\mathbb{C}_{U_2}) = H^*(U_2 \cap \overline{U_1}, U_2 \cap \partial\overline{U_1})$$

morphisms to be the relative de Rham cplx.

$$\mathrm{Open}(X) \xrightarrow{\cong} \mathrm{Morse}(X) \xrightarrow{\Phi} \mathrm{Fuk}(T^*X).$$

Morse A_∞-cat:

objects: (U, f)

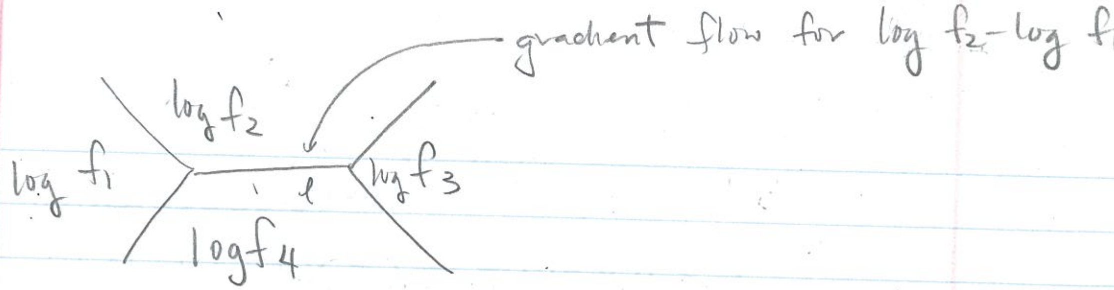


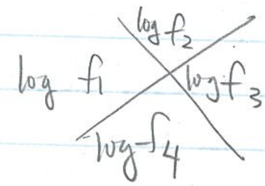
Fuk:

moduli of pseudohol polygons.

Morse:

moduli of trees w/ gradient flow





Morse: $\mathrm{Hom}((V_1,f_1),(V_2,f_2))$ critical pts of $\log f_2 - \log f_1$ on $V_1 \cap V_2$

A_∞ operations analogous to Fuk.
(differential, composition, ...)

$\mathrm{Morse}(X) \cong \mathrm{Open}(X)$

For an individual hom set,
Morse theory for $\log f_2 - \log f_1$ on $U_2 \cap U_1$.

Actual proof:

deg i currents on X
(roughly: distribution-valued
i-forms,
dual to $\dim X - i$ forms,
all done in relative setting)

$\mathrm{de\,Rham}(X) \cong \mathrm{Current}(X)$

$\mathrm{cochain}(X) \longrightarrow \mathrm{Current}(X)$

"idempotent" on $\mathrm{Current}(X)$
image canonically isomorphic to $\mathrm{Morse}(X)$.

Have $\Omega(X) \longrightarrow \mathrm{Current}(X)$
defined using a kernel on $X \times X$ built from Morse ftn.

$\sum [Y_+] \boxtimes [Y_-]$ (stable/unstable set)

Fuk(T^*M)

objects: exact Lagrangians that behave well at ∞.

ball compactification

fibers 

require Lagrangians to have well-behaved closure
(look "straight" at ∞)

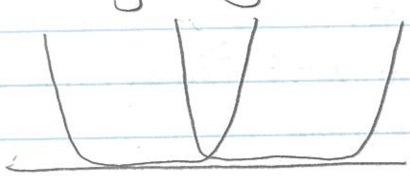
Fuk(T^*M) — noncpt moduli space

perturb at ∞
via a Hamiltonian (normalized geodesic) flow.
extends to the ball compactification

$\Rightarrow$ make everything is disjoint at ∞.

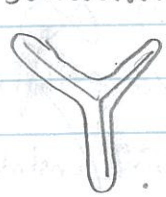
Compute $\mathrm{Hom}_{Fuk}(\Phi(V_1, f_1), \Phi(V_2, f_2))$
A_∞-ops

1. Perturb at ∞.
2. Dilate so that except at the boundary, everything is very close to the zero section.



Now, moduli spaces in A_∞ structure are diffeomorphic to the moduli spaces of gradient lines

Pseudoholom. curves "looks 1-D."



$\hookrightarrow$ look like gradient flow lines.

(start w/ something approximately pseudoholom)

Why equivalence?

$\mathrm{Id}: F(T^*X) \to F(T^*X)$

convolution w/ $T^*_\Delta(X \times X)$ in $T^*X \times T^*X$

(after reversing direction in first T^*X $(x,\xi) \mapsto (x,-\xi)$)

locally triangulate diagonal and deform.

$T^*_{\Delta_t}(X \times X)$ (Check : preserves homs, A_∞-ops)

horizontal $\Delta_t \subset X \times X$.

Then correspondence $\rightsquigarrow \sum [\ ] \cdot [\ ]$