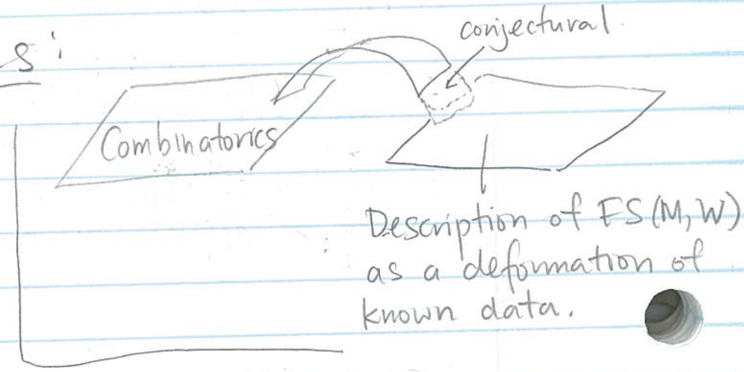
12/2.

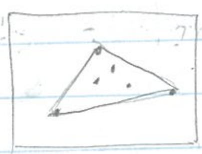
Exercise. Let L be a finitely-generated cochain complex. Show that a (deg 1) derivation
$d: \mathrm{Sym}(L^\vee[1]) \circlearrowleft$ s.t. $d^2 = 0$
gives L an L_∞-algebra structure.

Today: Kapranov-Kontsevich-Soibelman (1408, ...)

Some combinatorics:



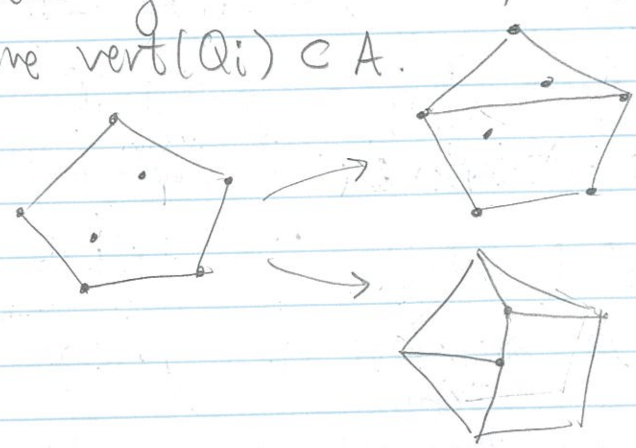
Fix $d \geq 1$, and a collection of finite points $A \subset \mathbb{R}^d$ in general position.
Assume $|A| \geq d+1$

d=2 

Let Q = convex hull of A (a d-simplex).

Defn. A polyhedral decomposition of (Q, A) is what you think it is, i.e., write $Q = \bigcup_i Q_i$ where $\mathrm{vert}(Q_i) \subset A$.

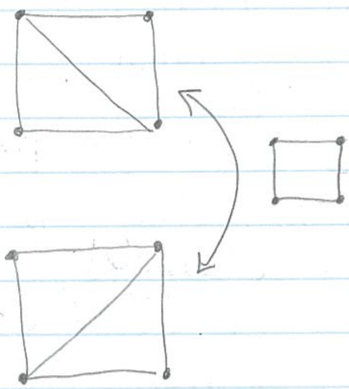
convex polyhedra

e.g. 

(⚠ Need to consider "regular" poly. decomp. — see KKS.)

Let $\Sigma(A)$ denote the collection of all polyhedral decompositions of (Q, A).
This determines a cell complex!

Vertices: Triangulations w/ no leftover dots.
Edges: Flipping triangles.



In general:
poset structure by refinement
$\rightsquigarrow$ cell complex.

Observation: Any face of a cell in $\Sigma(A)$ is a product of cells in $\Sigma(A_i)$ for $\{A_i \subset A\}$

<u>Lemma.</u> $\forall A$, these combinatorics determine an L_∞-algebra.

Cultural rmks: "L" ~ Lie
$\rightsquigarrow$ all Lie algebras are L_∞.
all <u>dg Lie algebras</u> are L_∞
Lie alg object in Chains.

i.e. data of
- L a gr. vector spaces
- $l_1 : L \to L[1]$ degree 1, s.t. $l_1^2 = 0$
- $l_2 : L \otimes L \to L$, $x \otimes y \mapsto [x,y]$

such that
- l_1 is a derivation for l_2: $l_1[x,y] = [l_1 x, y] + (-1)^{|x|}[x, l_1(y)]$.
- $[y,x] = (-1)^{1+|x||y|}[x,y]$
- $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$.

Ex. Fix cpt mfld M, trivial G-bundle $P \to M$.
Fix a flat connection A on P

$$\rightsquigarrow \Omega^{\bullet}(\mathrm{ad}\, P) \circlearrowleft (d_{dR} + A \wedge -)$$

"ad P-valued forms"

A flat $\Longrightarrow (d + A \wedge -)^2 = 0$

If $\alpha \otimes v \in \Omega \otimes \mathfrak{g}$,

$$[\alpha \otimes v, \alpha' \otimes v'] = (-1)^{|\alpha'||v|} \alpha \wedge \alpha' \otimes [v, v'] \in \Omega \otimes \mathfrak{g}$$

Rmk. This dgla is a model for the formal tangent bundles of {flat cnxns} at A.

Roughly, a L_∞-alg is a Lie algebra where the Jacobi identity only holds up to (specified) homotopy.

Pf of Lemma.

Obs./Defn. (exercise)
Fix V a graded vector space (f.d.), $V = \bigoplus_{i \in \mathbb{Z}} V^i$
Then can make

$\mathrm{Sym}(V) \leftarrow$ free graded commutative algebra generated by V

$$\| $$

$$k \oplus V \oplus \mathrm{Sym}^2(V^{even}) \oplus \Lambda^2(V^{odd}) \oplus \cdots$$

Example $V = V^0 = \mathrm{span}\langle x_1, \dots, x_k \rangle$
then $\mathrm{Sym}(V) = \mathbb{C}[x_1, \dots, x_k]$.

then the data of a derivation

$$d : \mathrm{Sym}\, V \longrightarrow \mathrm{Sym}\, V[1]$$

$$d(\alpha\beta) = (d\alpha)\beta + (-1)^{|\alpha|}\alpha\, d\beta$$

s.t. $d^2 = 0$ and $d(\mathrm{Sym}^{\geq 1} V) \subset \mathrm{Sym}^{\geq 1} V$.

is equivalent to the data of an L_∞-algebra on $V^\vee[-1]$.

Why? Since d is a derivation,

$\mathrm{Sym}\, V \xrightarrow{d} \mathrm{Sym}\, V$ is determined by

$$\begin{tikzcd} V \subset \mathrm{Sym}\, V \arrow[r, bend right] & \mathrm{Sym}\, V \end{tikzcd}$$ (by Leibniz rule).

e.g. $d(v \cdot w) = dv \cdot w + (-1)^{|v|} v \cdot dw$.

Ex. $V \longrightarrow \mathrm{Sym}\, V$

$$\begin{aligned} \mathrm{Sym}\, V &= k \\ &\oplus\ \mathrm{Sym}^1 V \quad (\text{via } l_1^\vee) \\ &\oplus\ \mathrm{Sym}^2 V = \mathrm{Sym}^2(V^{even}) \oplus \Lambda^2(V^{odd}), \quad (\text{via } l_2^\vee) \\ &\oplus \\ &\ \ \vdots \\ &\oplus\ \mathrm{Sym}^k V \quad (\text{via } l_k^\vee) \end{aligned}$$

$\rightsquigarrow$ get $V^\vee \xleftarrow{l_k} (\mathrm{Sym}^k V)^\vee$

Using $d^2 = 0$, get L_∞-relations.

This is a main example of <u>Koszul duality</u>

Comm $\longleftrightarrow$ Lie, Ass $\longleftrightarrow$ Ass.

Need to exhibit a deg 1 derivation on Sym V for some V.

Take $V = C^{cell}_{-*}(\Sigma(A))_{\circlearrowleft d}$.

We remarked that the ∂ of any face of $\Sigma(A)$ is a product of lesser faces.

$$d : V \longrightarrow \mathrm{Sym}\, V$$

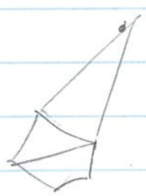
$$\sigma \in \Sigma(A) \qquad \sigma \longmapsto \partial\sigma = \prod \sigma'$$

Since d is a differential for a cell complex $d^2 = 0$. Extend to $\mathrm{Sym}\, V \to \mathrm{Sym}\, V$ by Leibniz rule. $\square$

Given $A \subset \mathbb{R}^d$, choose some point "∞" $\in \mathbb{R}^d$ far away from A and s.t. $\tilde{A} = A \cup \{\infty\}$ is in general position.

Let $\mathfrak{g}$ denote the L_∞-alg associated to A.
— $\tilde{\mathfrak{g}}$ —— " ————— $\tilde{A}$.

Then $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ is a sub L_∞-algebra.



Consider those subpolyhedra of $\tilde{Q}$ that contain ∞ as a vertex.

This defines an ideal $\mathfrak{g}_\infty \subset \tilde{\mathfrak{g}}$.

$\rightsquigarrow \tilde{\mathfrak{g}} \simeq \mathfrak{g}_\infty \rtimes \mathfrak{g}$.

$\rightsquigarrow$ action of $\mathfrak{g}$ on $\mathfrak{g}_\infty$.

For $d=2$. (think $\mathbb{R}^2 = \mathbb{C}$ = target of some Lefschetz fibration $W: M \to \mathbb{C}$)

When we defined $V \xrightarrow{d} \mathrm{Sym}\, V$
for $V = C_*(\Sigma(A))$, can lift to

$$V \xrightarrow{d} T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$$

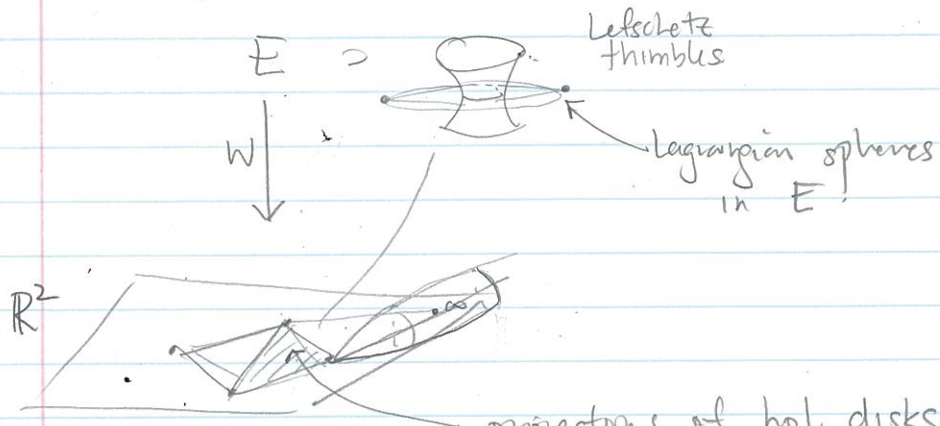
free associative algebra
(remembering order of subdivisions)

$\rightsquigarrow$ lift the L_∞-alg $\mathfrak{g}_\infty$ to an A_∞-alg R.
(roughly: "$[x,y] = xy - yx$" for some A_∞-alg.)

The map $\mathfrak{g} \to \mathrm{Der}(\mathfrak{g}_\infty)$
lifts to $\mathfrak{g} \longrightarrow CC^*(R)[\pm 1]$ — a map of L_∞-alg
Hochschild cplx of R

<u>Thm.</u> (KKS) This is an equivalence of L_∞-algebras.

(so $\mathfrak{g} \longleftrightarrow$ deformations of R)



$A = \mathrm{CntVal}(w)$

Idea: $\exists$ version of $\mathfrak{g}, \mathfrak{g}_\infty, R$. taking coefficients coefficients in $\mathcal{L}$, i.e. $C_*(\Sigma(A), \mathcal{L})$.
and the algebra R has $CC^*(R) \simeq \mathfrak{g}_W$.

All we need is to produce a nice element (Maurer-Cartan el't) of $\mathfrak{g}_W$ to deform R.

Conjecture: $FS(E, W) \simeq Mod_{\tilde{R}}$
deformation of R.

(obtained by counting holomorphic disks including ∞)