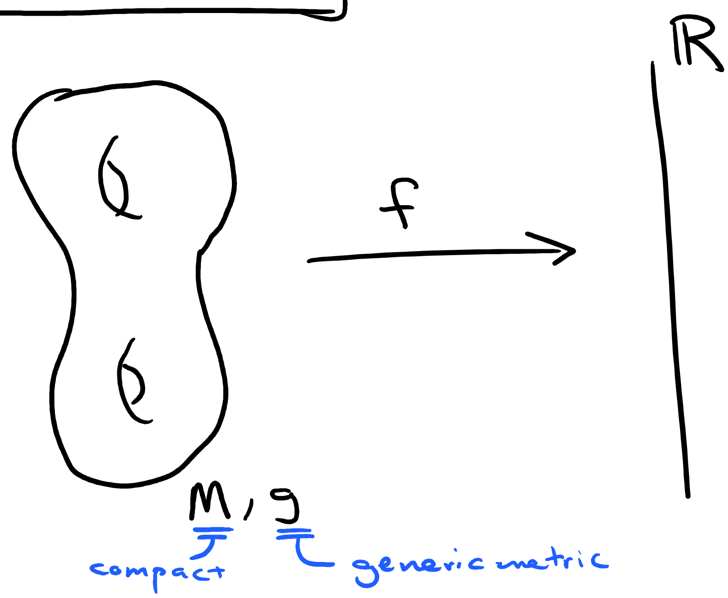
DGFT Lecture 1 - Basics of Floer Theory

Friday, September 23, 2016 2:01 PM

(Boyu Zhang speaking)

§ Morse Theory



Suppose f is Morse, i.e. $\text{Hess} f$ is non-deg. at critical points

For any crit pt,

index = # of negative eigenvalues of $\text{Hess} f$

Set $C_n := \mathbb{Z}^{\{\text{crit pts of index } n\}}$

Can define

$$\partial_n : C_n \longrightarrow C_{n-1}$$

$$p \longmapsto \sum_q \#\left(\begin{array}{c}\text{gradient flows } p \text{ to } q \\ \text{counted w/ orientation}\end{array}\right) \cdot q$$

Some things to prove:

(1) #(flow lines) is finite

(2) $\partial^2 = 0 \rightsquigarrow$ get homology

(3) Homology independent of metric g

(4) Homology independent of Morse function f

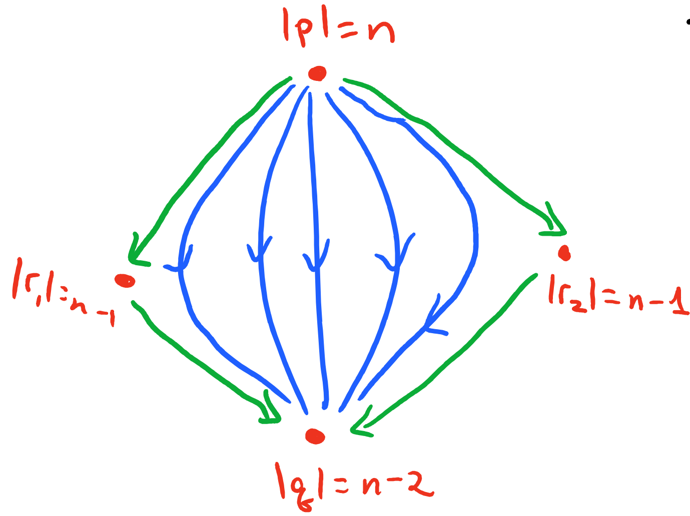
For (1):

- Impose Morse-Smale condition
- prove compactness of space of flow lines

For (2):

- Consider flow lines C_n to C_{n-2}. This is 1-D mfld. Can compactify by including broken trajectories which form the bdry.

$|p|=n$ These broken trajectories



These broken trajectories count coefficients of ∂^2. Need a technical gluing argument

For (3) + (4): Continuation arguments (won't go into today)

§ Floer Theory

(M,ω) = symplectic mfld

$H(t)$ = time dependent family of fns on M (Hamiltonian)

Hamiltonian Vector field $X_H(t)$ defined by $i_{X_H(t)}\omega = dH(t)$

Arnold Conjecture: About Period 1 orbits of $X_H(t)$

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Say $H(t)$ independent of t and Morse.

Then $X_H = 0$ at crit pts, so every crit point of H gives a period 1 orbit.

Taking $b_i(M) = \operatorname{rank} H_i(M)$ where

$$H_i(M) = \ker(\partial_i) / \operatorname{im}(\partial_{i+1})$$

then

$$b_i \leq \operatorname{rank}(C_i)$$
$$= \#\text{ crit pts of index } i$$

$$\Longrightarrow \sum b_i \leq \#\text{ crit pts}$$ "Morse Inequality"

$$\boxed{\begin{array}{l}\textbf{Arnold Conjecture: } \text{For general } H(t), \\ \text{if all period 1 orbits are non-degenerate} \\ \text{then} \\ \qquad \#\text{orbits} \geq \sum b_i\end{array}}$$

Floer's approach:

Assume $\pi_2(M) = 0$

$$\mathcal{A}_H : \underset{\text{contractible}}{\mathrm{Map}_0(S^1, M)} \longrightarrow \mathbb{R}$$

$$u \longmapsto -\int_D \tilde{u}^* \omega + \int_0^1 H_t(x(t))\,dt$$

$\tilde{u} : D \to M$ s.t. $\tilde{u}|_{\partial D} = x$

well-defined since $\pi_2(M) = 0$

Crit pts of $\mathcal{A}_H \Longleftrightarrow$ Period-1 orbits

So, if we can generalize Morse theory to $(\mathrm{Map}_0(S^1, M) ; \mathcal{A}_H)$, then we will have a Morse inequality that gives a lower bound on the # of orbits!

Issues

(1) Use Sobolev spaces to make all the spaces into Banach mflds and

spaces into Banach mf'lds and
$\mathcal{A}_H$ smooth.

(2) Flow line picture doesn't really work.

Look at PDEs directly

"flowlines" are $S^1 \times \mathbb{R} \longrightarrow M$

satisfying an equation whose first order term is $\bar{\partial}$

(3) Compactness issue is understood by Gromov's compactness package

(4) Non-degeneracy of critical points

(5) Gluing argument

(6) Regularity of moduli space of trajectories

Actually

problematic!

$$\begin{tikzcd} V \arrow[d] \\ \mathrm{Map}_u(S^1\times\mathbb{R}, M) \arrow[u, bend right, "\substack{\text{section } F \\ \bar\partial + \text{lower order term}}"'] \end{tikzcd}$$

Vector bundle...

$$V = \Omega^{0,1}(S^1\times\mathbb{R}, u^*TM)$$

Elliptic regularity for $F = \bar\partial +$ lower order

In general, however, F doesn't have to be surjective.

Freed–Uhlenbeck:

perturbation space (of almost complex structs, in our case)

$$\tilde F : P \times \mathrm{Map}_u(S^1\times\mathbb{R}, M) \longrightarrow V$$

$$(p, \varphi) \longmapsto \bar\partial_p \varphi + \dots$$

Suppose P large enough s.t.

$$\pi_{\text{fibers}} \circ \tilde{F}$$

has surjective tangent map,

$S = F^{-1}(0\text{-section})$ is an ∞-dim'l submfld of $P \times \mathrm{Map}_u(S^1 \times \mathbb{R}, M)$.

Project S to P. $\exists p_0 \in P$ s.t. $\pi^{-1}(p_0)$ is regular in S. Solution of $\bar{\partial}_{p_0}^{-1}$ is going to be $\pi^{-1}(p_0)$, which is regular

Problems w/ this approach

(1) Need somewhere injectivity

(2) If M Kähler, probably want to keep

this condition.

(3) Similarly if M has symmetry