Boyu - Transversality issues in Floer Theory

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1 Morse Theory from Floer-ish Prospective

Definition 1. Let M be a compact manifold, and g a generic metric on M. A function $f : M \to \mathbb{R}$ is *Morse* if $Hessf$ is nondegenerate at critical points.

For any critical point $p \in M$ of f, the *index* of p is the number of negative eigenvalues of $Hessf$.

Definition 2. (Morse homology)

For each $n \geq 0$, let C_n be the free abelian group generated by the critical points of index n.

The boundary map $\partial_n : C_n \to C_{n-1}$ takes $p \mapsto \sum_q \#(\text{gradient flow lines from } p \text{ to } q) \cdot q$.

Some concerns for well-definition:

1. #(flow lines) is finite. Morse-Smale condition: f is Morse, flow lines intersect transversely. Then show the moduli space of flow lines is compact.
2. $\partial^2 = 0$. Here, consider flow lines from index $n$point p to an index $n-2$ point q. A 1-parameter family of flow lines converges to either a flow line p to q, or a broken flow line (concatenation of flow line from index n to $n-1$ to $n-2$). We need to show that only broken flow lines are the limits, and that the two broken flow lines bounding the 1-dimensional family of flow lines are distinct. It suffices to argue that for each broken flow line, there is a unique way to deform it to a smooth trajectory. We use a technical gluing property.
3. Homology is independent of our choice of metric. Cobordism argument, construct isomorphism between chains.
4. Homology is independent of choice of Morse function f.

Remark. Recall a Morse function for M gives a CW-decomposition. Each n-cell is the descending manifold for an index n critical point. This approach does not generalize to the infinite case.

2 Floer Theory

Here we use Hamiltonian Floer theory as an example.

Let (M, ω) be a symplectic manifold, $H(t)$ a time-dependent Hamiltonian function. Each $H(t) : M \to \mathbb{R}$. (Consider $H(t)$ 1-periodic.)

Define $X_H(t)$ such that $i_{X_H(t)}\omega = dH(t)$.

2.1 Arnold Conjecture

How do we count period 1 orbits of the Hamiltonian vector field $X_H(t)$ on M?

Suppose $H(t)$ were independent of t and Morse, then $X_H = 0$ at critical points. Hence every critical point of H gives a period 1 orbit. By the Morse inequality, this implies that the number of period 1 orbits is $\geq$ the sum of the Betti numbers (coming from Morse homology).

Conjecture For general $H(t)$, if all period 1 orbits are nondegenerate (nonstationary), then # orbits $\geq \sum b_i$.

This was Floer's motivation for defining his theory.

2.2 Floer's approach

Assume $\pi_2(M) = 0$. Consider the infinite-dimensional space $Map_0(S^1, M)$, and a functional

$$\mathcal{A}_H : Map(S^1, M) \to \mathbb{R}$$

$$\mathcal{A}_H(x) = -\int_D u^*\omega + \int_0^1 H_t(x(t))\, dt$$

where $u : D \to M$ is a map of a disk to M.

Then x is a critical point of $\mathcal{A}_H$ iff x is a period 1-orbit.

Floer generalized Morse theory to be used on $(Map_0(S^1, M), \mathcal{A}_H)$, to obtain a Morse inequality giving a lower bound on the number of orbits.

Issues:

1. We want to understand $Map_0(S^1, M)$ as a Banach manifold structure (for Fredholm theory later). Use Sobolev spaces to make all the spaces Banach manifold with $\mathcal{A}_H$ smooth.
2. The gradient flow lines will not be unique; we get PDEs. We look at the PDEs directly. Here, the "flow lines" are J-holomorphic cylinders $S^1 \times \mathbb{R} \to M$ (satisfying an equation whose first order term is $\bar{\partial}$).
3. The compactness issue is understood via Gromov's compactness package.
4. Nondegeneracy of critical points.
5. Gluing argument. Again, technical analysis to confirm that trajectories are not double-counted.
6. (When $\dim M \geq 8$) Regularity of moduli space trajectories; they need to be cut up by transverse maps.

There is a map $Map(S^1 \times \mathbb{R}, M) \to \mathcal{V}$ whose first order term is $\bar{\partial}$. Here $\mathcal{V}$ is an infinite rank vector bundle over the infinite dimensional space $Map(S^1 \times \mathbb{R}, M)$, with fibers $\Omega^{0,1}(S^1 \times \mathbb{R}, u^*(TM))$. It turns out the solutions to the PDE is the pullback by this map of the zero section of $\mathcal{V}$.

The tangent map of F projected to the fiber direction gives a Fredholm operator, whose index can be computed by Riemann-Roch. In general, this operator need not be surjective; in this case, the zero-section of $\mathcal{V}$ is not regular. We need to perturb our equation.

2.3 Freed-Uhlenbeck

Consider a Banach space P serving as a perturbation space. Suppose we could extend our fibers F to

$$\tilde{F} : P \times Map_u(S^1 \times \mathbb{R}, M) \to \mathcal{V}$$

$$(P, \varphi) \mapsto \bar{\partial}_p \varphi + \dots$$

Suppose P is large enough such that $\pi_{fibers} \circ \tilde{F}$ has a surjective tangent map. Let $S = F^{-1}(\text{ 0 section })$ is an infinite dimensional submanifold of $P \times Map_u(S^1 \times \mathbb{R}, M)$. Now project S to P. There exists $p_0 \in P$ such that $\pi^{-1}(p_0)$ is regular. Then the solution of $\bar{\partial}_{p_0}^{-1}$ is going to be $\pi^{-1}(p_0)$, which is regular.

Problems with this approach:

1. Need $S^1 \times \mathbb{R}$ to be somewhere injective.
2. Then we don't want to perturb our J-holomorphic structure when M is Kahler.
3. If M has symmetry, we want our perturbations to be equivariant. (For example, branched covers.)