DGFT Lecture 2 - Tor; Serre's intersection formula

Friday, September 30, 2016 2:03 PM

(Melissa Zhang speaking)

X = ambient space
$U, V \subset X$
$W = U \cap V$

§. Serre's Intersection Formula

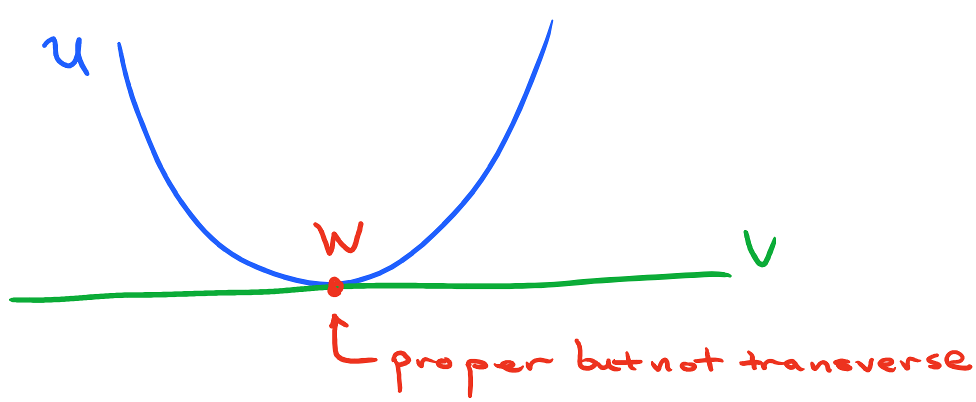
(see e.g. Local Algebra)

Hypotheses:
- Smooth affine varieties
- Intersect properly

$$\dim U + \dim V = \dim X + \dim W$$

e.g. $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x,y])$

$U = V(Y - X^2)$ $V = V(Y)$



Recall: $\otimes$ is $\cap$:

For $\mathcal{U}, V \subset \mathbb{A}^n$
$R = \mathbb{C}[\mathbb{A}^n]$

$$I := \mathcal{I}(\mathcal{U}), \quad J = \mathcal{I}(V)$$

$$\mathcal{U} \cap V = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supset I, \ \mathfrak{p} \supset J\}$$
$$\cong \operatorname{Spec}\left(R/_{I+J}\right)$$
$$\cong \operatorname{Spec}\left(R/_{I} \otimes_R R/_{J}\right)$$

<u>Review</u>: Tor

clever notation:
r1ng

R = commutative ring with 1

A, M = R-module

$- \otimes_R A$ is right exact

Tor $\longleftrightarrow$ Left-derived functor

i.e. to compute $\operatorname{Tor}^R_\bullet(A, M)$

1. Projective resolution
$$\cdots \to P_1 \to P_0 \to 0$$

2. Apply $A \otimes_R -$
$$\cdots \to A \otimes_R P_1 \to A \otimes_R P_0 \to 0$$

3. Take homology

Reducing to the diagonal:

$$\mathcal{U}, \mathcal{V} \subset \mathbb{A}^n, \qquad R = \mathbb{C}[\mathbb{A}^n]$$

$$\mathcal{U} \cap \mathcal{V} = (\mathcal{U} \times \mathcal{V}) \cap \Delta \subset \mathbb{A}^n \times \mathbb{A}^n$$

$$R/p_{\mathcal{U}} \otimes_R R/p_{\mathcal{V}} = \left(R/p_{\mathcal{U}} \otimes_{\mathbb{C}} R/p_{\mathcal{V}} \right) \otimes \left(\frac{R \otimes_{\mathbb{C}} R}{\mathfrak{d}} \right) \qquad \cong R$$

where $\mathfrak{d} = \mathcal{I}(\Delta) = \langle 1 \otimes x - x \otimes 1 \rangle$

$$\mathrm{Tor}_{\bullet}^{R}\left(R/p_{\mathcal{U}}, R/p_{\mathcal{V}}\right) \cong \mathrm{Tor}_{\bullet}^{R \otimes_{\mathbb{C}} R}\left(\frac{R \otimes_{\mathbb{C}} R}{\mathfrak{d}}, R/p_{\mathcal{U}} \otimes_R R/p_{\mathcal{V}} \right)$$

Serre's Intersection Formula

Let R = coord ring of X

Hypotheses:

- $\mathcal{U}, \mathcal{V}$ irred varieties in X
- W irred component of $\mathcal{U} \cap \mathcal{V}$
- $\mathcal{U}, \mathcal{V}$ intersect properly at W
- W meets the smooth pts of X

$$\left(\underline{\text{i.e.}} \qquad A := R_{p_W} \right)$$

Then

$\dim X$

Then

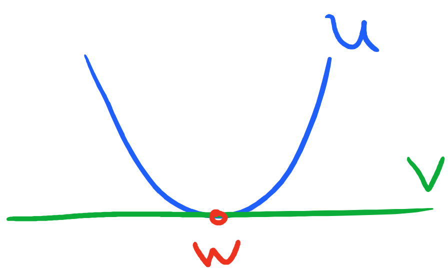
$$\chi = \sum_{i=0}^{\dim X} (-1)^i \, \ell_A \, \mathrm{Tor}_i^A\left(A/p_u, A/p_v\right)$$

$$\left(\text{when } W = pt: \quad \chi = \sum_{i=0}^{\dim X} (-1)^i \dim_{\mathbb{C}} \mathrm{Tor}_i^R\left(R/p_u, R/p_v\right) \right)$$

Thm [Serre]

- If U, V do not intersect properly at W, then $\chi = 0$
- Otherwise $\chi > 0$ and is equal to the intersection multiplicity at W

e.g.



$R = \mathbb{C}[x,y]$

$p_u = (Y - X^2)$ $\qquad$ $p_v = (Y)$

1. $0 \longrightarrow R \xrightarrow{Y} R \longrightarrow 0$ $\longleftarrow$ resolution for $R/p_v = \mathbb{C}[x,y]/Y$

2. $0 \longrightarrow R/p_u \otimes_R R \xrightarrow{1 \otimes Y} R/p_u \otimes_R R \longrightarrow 0$

$0 \longrightarrow R/p_u \xrightarrow{y} R/p_u \longrightarrow 0$

$$0 \longrightarrow \mathbb{C}[x] \xrightarrow{x^2} \mathbb{C}[x] \longrightarrow 0$$

3.

$$h_1 = 0/0 \qquad h_0 = \mathbb{C}[x]/x^2$$

$$\mathrm{Tor}_1 = 0 \qquad \mathrm{Tor}_0 = \mathbb{C} + \mathbb{C}x$$

$$\dim = 0 \qquad \dim = 2$$

$$\Longrightarrow \quad [u]\cdot[V] = 2$$

Had a discussion about trying to find an example where higher Tor groups not all zero, but couldn't find one.