DGFT Lecture pre-3 - Ho(co)limits

Friday, September 30, 2016 2:45 PM

(Ian Banfield speaking)

§1. Colimit

Defn: A colimit $\varinjlim D$ for a functor $D: \mathcal{J} \to \mathcal{C}$ is a universal arrow

index category

from D to the diagonal functor

$$\Delta: \mathcal{C} \longrightarrow \mathcal{C}^{\mathcal{J}}$$

Functor category
Objects: $F: \mathcal{J} \to \mathcal{C}$
Morphisms: Nat. Transf

$\underset{\mathcal{C}}{\overset{\Uparrow}{\Delta r}}$ = "constant diagonal map": $\forall j$, $\Delta r(j) = r$

i.e. a pair (r, u), $u: D \longrightarrow \Delta r$

"$\varinjlim D$"

$r \in \mathrm{Ob}\,\mathcal{C}$, $u \in \mathrm{Mor}(\mathcal{C}^{\mathcal{J}}, \mathcal{C}^{\mathcal{J}})$

s.t. $\forall$ morphisms $D \xrightarrow{f} \Delta s$

$\exists$ a map $f': r \to s$

such that this commutes

$$\begin{tikzcd} D \arrow[r, "u"] \arrow[d, equal] & \Delta r \arrow[d, "\Delta f'"] \\ D \arrow[r, "f"] & \Delta s \end{tikzcd}$$

Q: What does this mean?

$$
\begin{tikzcd}
D(i) \arrow[r, "D(f)"] \arrow[dr, "u_i"'] \arrow[ddr, bend right] & D(j) \arrow[r] \arrow[d, "u_j"] \arrow[dd, bend right] & D(k) \arrow[dl, "u_k"] \arrow[ddl, bend left] \\
& \varinjlim D \arrow[d] & \\
& S &
\end{tikzcd}
$$

If S exists like this, then $\exists!$ green arrow exists

e.g. $\mathcal{C} = \mathrm{Top}$

$J = 1 \leftarrow 0 \rightarrow 2$

$$
\begin{tikzcd}
* \arrow[dr] & S^n \arrow[l] \arrow[r, hook, "i"] \arrow[d] & D^{n+1} \arrow[dl] \\
& \varinjlim D = S^{n+1} &
\end{tikzcd}
$$

where $D = * \leftarrow S^n \xhookrightarrow{i} D^{n+1}$

$$\left(\text{Think } \varinjlim D = {* \sqcup S^n \sqcup D^{n+1}} \Big/ {S^n \sim * \sim i(S^n \hookrightarrow D^{n+1})} \right)$$

e.g.: $D' = * \longleftarrow S^n \longrightarrow *$

$$
\begin{tikzcd}
* \arrow[dr] & S^n \arrow[l] \arrow[r] \arrow[d] & * \arrow[dl] \\
& \varinjlim D' = * &
\end{tikzcd}
$$

$\varinjlim D = *$

(Think $* \sqcup S^n \sqcup *' \big/ * = S^n = *'$)

Given $\eta : D \to D'$ natural transf.

for $i \in J$, $\eta_i : \underset{\text{Top}}{\underset{\cap}{D(i)}} \longrightarrow \underset{\text{Top}}{\underset{\cap}{D'(i)}}$

<u>Fact</u>: If we have a natural transf

$$\eta : D \longrightarrow D'$$

s.t. η_i are homeomorphisms

then $\varinjlim D = \varinjlim D'$

In our example:

Have weak equivalence $D \to D'$

<u>BUT</u> $\varinjlim D \neq \varinjlim D'$

so this is htpy-theoretically <u>BAD</u>

$*$ Motivates introduction of htpy-theoretic version