DGFT Lecture 3 - Ho(co)limits

Friday, October 7, 2016 2:02 PM

(Ian Banfield speaking)

Last time:

In Top:

$$\begin{tikzcd}
* \arrow[d, equal] & S^n \arrow[l] \arrow[r] \arrow[d, equal] & D^{n+1} \arrow[d, "\simeq"] & = D \\
* & S^n \arrow[l] \arrow[r] & * & = D'
\end{tikzcd}$$

$$\left.\begin{array}{l} \operatorname{colim}(D) = S^{n+1} \\ \operatorname{colim}(D') = * \end{array}\right\} \text{Not weakly equivalent!}$$

Homotopy colimit will be a colimit that preserves weak equivalence

How to define:

1. Simplicial objects X
2. Geometric Realization of X — "How to think of X as CW complex"
3. Simplicial replacement — "How to think of category as simplicial object"

1. Simplicial Objects

Defn: Simplex category Δ consists of

- Objects $[n] = [0 \to 1 \to \cdots \to n]$

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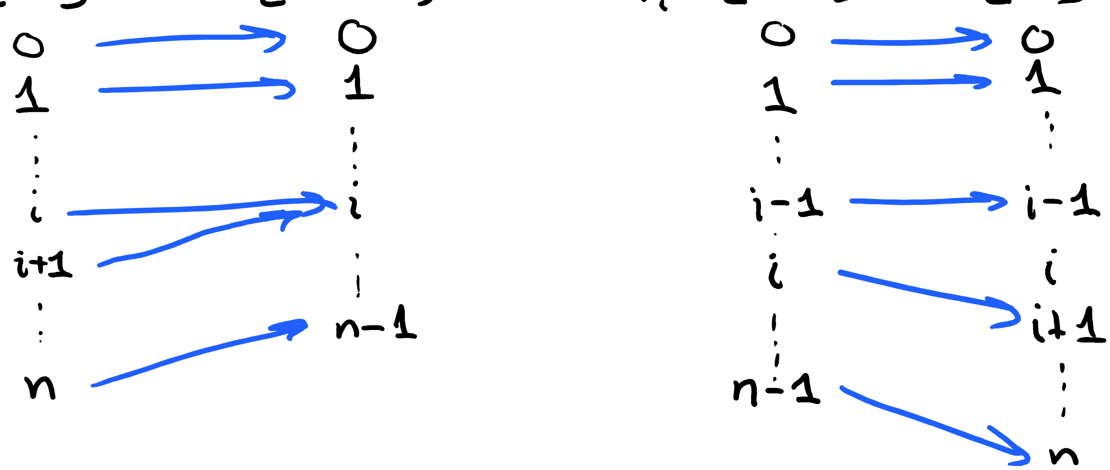
- Morphisms $[n] \to [m]$ order-preserving maps of sets

e.g.

Have maps

$s_i : [n] \longrightarrow [n-1]$ $\qquad\qquad$ $d_i : [n-1] \longrightarrow [n]$



Defn: A simplicial object is

$$X : \Delta^{op} \longrightarrow \mathcal{C}$$

Fact: Can describe simplicial object as follows

- $X_0 = X([0])$
- $X_n = X([n])$
- Face maps $\quad d_i : X_n \longrightarrow X_{n-1}$
- Degeneracy maps $\quad s_i : X_{n-1} \longrightarrow X_n$

- Lots of Stuff
- $$\begin{tikzcd} X_n \arrow[d, equal] \arrow[r, "s_j"] \arrow[dr, phantom, "\square"] & X_{n+1} \arrow[dl, "d_j"] \\ X_n & \end{tikzcd}$$

Just for clarity:

X = simplicial object:

Face maps

$$X_3 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_1 \;\substack{\longrightarrow\\\longrightarrow}\; X_0$$

degeneracy maps

Cosimplicial object:

$X : \Delta \longrightarrow \mathcal{C}$

Coface maps

$$X_0 \;\substack{\longrightarrow\\\longrightarrow}\; X_1 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; X_3$$

codeg maps

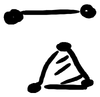
② The Geometric Realization

We can define a cosimp object

Δ^k = k-simplex in $\mathbb{R}^n$

$\Delta^0 = \bullet$

$\Delta^1 = \bullet\!\!-\!\!\bullet$

$\Delta^2 =$ 

Coface maps:

$d_i : \Delta^n \longrightarrow \Delta^{n+1}$

inclusion onto face opposite i

Codegeneracy maps: $s_i : \Delta^n \longrightarrow \Delta^{n-1}$

Projection onto face opposite i

For X simplicial object

$$X : \Delta^{op} \longrightarrow Top$$

the geometric realization is

$$|X| = \mathrm{coeq}\left(\coprod_{[n]\to[k]} X_k \times \Delta^n \overset{f}{\underset{g}{\rightrightarrows}} \coprod_i X_i \times \Delta^i \right)$$

$$= \coprod_i X_i \times \Delta^i \Big/ \sim$$

where

$$X_k \times \Delta^n \xrightarrow{f} X_n \times \Delta^n$$
$$X_k \times \Delta^n \xrightarrow{g} X_k \times \Delta^k$$

How to define f, g?

$$\alpha : [n] \to [k] \Longrightarrow \alpha^* : X_k \to X_n \text{ for } f$$
$$\Longrightarrow \alpha_* : \Delta^n \to \Delta^k \text{ for } g$$

That is $\sim$ given by

$$(x, \alpha_*(s)) \sim (\alpha^*(x), s)$$

Example: Let G = group. Take as simplicial object.

$X_0 = \{e\}$

$X_1 = G$ $\qquad\qquad s_i(g_1, \ldots, g_n)$

$$X_0 = \{e\}$$

$$X_1 = G \qquad s_i(g_1, \dots, g_n) = (g_1, \dots, g_i, e, \dots, g_n)$$

$$X_2 = G \times G$$

$$\vdots$$

$$X_n = G^n \qquad d_i(g_1, \dots, g_n) = (g_1, \dots, g_i g_{i+1}, \dots, g_n)$$

$\Longrightarrow$ $|X|$ = classifying space of G

Property:

- "You can throw away degeneracies"

  If θ = degeneracy/face maps corresponding to when we write it

  $$(\theta(x), s) = (x, \theta(s))$$

  If $x = \theta(x')$, can repeat.

  We can think of just nondegenerate elts

③ Simplicial Replacement

"How to think of category as simplicial object"

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

Vertices = Objects
Edges = arrows
Triangles = composition

Generalize this to the following

Simplicial replacement

Simplicial replacement

For diagram $D : J \to \mathrm{Top}$, we'll form a simplicial object $\mathrm{srep}(D)$ as follows:

Face maps

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \coprod_{j_2\to j_1\to j_0} D(j_2) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \coprod_{j_1\to j_0} D(j_1) \;\rightrightarrows\; \coprod_{j\in J} D(j)$$

E.g.

$$\begin{tikzcd} A \arrow[loop above] \arrow[r, "f"] \arrow[rr, bend right] & B \arrow[loop above] \arrow[r, "g"] & C \arrow[loop above] \end{tikzcd}$$

yields

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; A \sqcup B \sqcup C \sqcup A_f \sqcup B_g \sqcup A_{g\circ f} \;\rightrightarrows\; A \sqcup B \sqcup C$$

Degeneracy maps?

$$s_i : \underbrace{D(j)}_{\substack{\text{indexed by}\\ [j_n \to \cdots \to j_0]}} \xrightarrow{\ \mathrm{id}\ } \underbrace{D(j)}_{\substack{\text{indexed by}\\ [j_n \to \cdots \to j_i \to j_i \to \cdots \to j_0]}}$$

Degeneracy maps:

$$s_i : \underbrace{D(j)}_{\text{indexed by } [j_n \to \cdots \to j_0]} \xrightarrow{id} \underbrace{D(j)}_{\text{indexed by } [j_n \to \cdots \to j_i \to j_i \to \cdots \to j_0]}$$

Similar for face maps

Defn: $\mathrm{hocolim}(D) = |\mathrm{srep}(D)|$

Example: $Y \xleftarrow{f} A \xrightarrow{g} * \;=\; D$

Claim: $\mathrm{hocolim}(D) = C(f)$

① srep:

Degenerate stuff (ignorable) $\substack{\rightarrow\\ \rightarrow\\ \rightarrow}$ $\underbrace{* \sqcup A \sqcup Y}_{\text{also degenerate}}$ $\sqcup A_f \sqcup A_g$ $\rightrightarrows$ $* \sqcup A \sqcup Y$

$$A_f \sqcup A_g \overset{\alpha}{\underset{\beta}{\rightrightarrows}} * \sqcup A \sqcup Y$$

② Realization

$$\frac{(A_f \times I) \sqcup (A_g \times I) \sqcup (* \times A \times Y)}{\sim}$$

$\sim$: $f(a) \in Y \sim (a,1) \in A_f \times \Delta^1$

$g(a) = * \sim (a,1) \in A_g \times \Delta^1$

$a \in A \sim (a,0) \in A_f \times \Delta^1 \sim (a,0) \in A_g \times \Delta^1$

$\Longrightarrow$ Get mapping cone!