Jake Implicit Atlases, I.

[redacted]@math.harvard.edu

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Problem:

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$ fiber product of C^∞ mflds.

$$\begin{tikzcd} X\times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Ex $X \hookrightarrow Z \hookleftarrow Y$ inclusions $\Rightarrow$ $X \times_Z Y = X \cap Y$.

Need $\pitchfork$.

Ex
$$\begin{tikzcd} s^{-1}(0) \arrow[r] \arrow[d] & X \arrow[d, "s"] \\ * \arrow[r, "0"'] & E \end{tikzcd}$$
$s^{-1}(0)$ is smth sub ~~mfld~~ when s submersion @ 0

$f \pitchfork g \Rightarrow$ no problem. But 3 things can go wrong!

(1) $X \times_Z Y$ not a mfld. (eg. any closed subset can be 0 locus)

(2) mfld, but of wrong dimension.

eg: $\mathbb{R}_x \times_{\mathbb{R}^2} \mathbb{R}_x$ ——
$= \mathbb{R}$, codim 1

(3) $X \times_Z Y$ could be wrong mfld!

Ex:  zero locus of x^2.

$[X \times_Z Y]$ not invariant under perturbation?

Class in what? Cobordism ring, or $H_*(Z)$?

Could have two diagrams ~~where~~ which are related by perturb., but whose pullbacks aren't invariant, just like ~~colims~~ colims change. So we'll do derived pullback, just like hocolims solve the issue.

Two approaches: • Spivak/Lurie: $\mathrm{Tor}^{C^\infty(Z)}(C^\infty(X), C^\infty(Y))$
(more like alg geom)

• other approach: use charts more explicitly; locally model spaces on 0 sets of C^∞ fns.

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Though $\dim X_\alpha$, $\dim E_\alpha$ not invariant, their difference is.

Def build-up: X Haus space.

A chart is open $U_\alpha \subset X$ w/

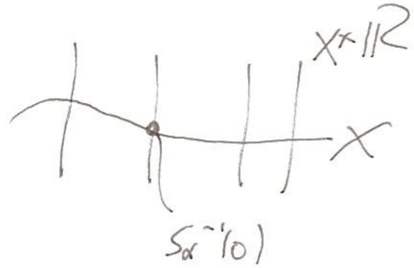
$(X_\alpha, E_\alpha, s_\alpha, \psi_\alpha)$

- X_α C^∞ mfld
- E_α vec sp fin dim
- $s_\alpha : X_\alpha \longrightarrow E_\alpha$ C^∞
- $\psi_\alpha : s_\alpha^{-1}(0) \underset{\text{homeo}}{\cong} U_\alpha$.

transition constant?

Consider

$(X_\alpha \times \mathbb{R},\ E_\alpha \times \mathbb{R},\ s_\alpha \times id_{\mathbb{R}})$



same zero set.

Defn The virtual dimension of U_α is

$$\mathrm{vdim}(U_\alpha) := \dim(X_\alpha) - \dim(E_\alpha).$$

More genlly,

$(X_\alpha \times E,\ E_\alpha \times E,\ s_\alpha \times f)$

$f : E \to E$ subm. @ 0.

An atlas has a lot of data to carry around for double, triple, etc overlaps.

Rmk Pardon uses Γ, more orbifolds; and has ∂. Ignore for now.

Defn (Pardon 3.1.1) X Hauss space. An implicit atlas on X of v.dim d on X is:

data of

- index set $A = \{\alpha\}$

(a) $\forall \alpha$, E_α fin dim vec space

(b) $\forall$ $I \subset A$ finite,

X_I Haus space

think of $X_{\alpha\beta}$ as trans. data btwn U_α, U_β, $X_{\alpha\beta\gamma}$ is etc.

st $X_\emptyset = X$

(c) $\forall$ $\alpha \in I \underset{\text{finite}}{\subset} A$ a map

$s_\alpha : X_I \longrightarrow E_\alpha$

$s_\alpha \equiv 0$ removes α from I.

So $s_\alpha^{-1}(0) \subset X_I$ is like $X_{I \setminus \alpha}$.

Notation: Given $I \subseteq J$,

$$s_I : X_J \longrightarrow E_I = \bigoplus_{\alpha \in I} E_\alpha$$

(d) (like transition fns)

$\forall\ I \subseteq J$, a homeo

$$\psi_{IJ} : \left(S_{J\setminus I}\big|_{X_J}\right)^{-1}(0) \cong U_{IJ}$$

where $U_{IJ} \subset X_I$ is open.

So cutting out $\beta \in J \setminus I$ from X_J yields open subset of X_I.

(e) Regular locus

$$X_I^{reg} \subseteq X_I \quad \underline{\text{open}},$$

a smth mfld, s_α smth on X_I^{reg}.

- "The part that really matters." Would like to set $X_I^{reg} = X_I$, but easier to have X_I sitting around in examples.
- smth mfld str __is__ data.

__Rmk__ Every $x \in X$ is cut out by smth data on some smth X_I^{reg}.

__Rmk__ Obvi, need $X_I^{reg} \neq \emptyset$ for some I; we'll see how this follows from compatibilities

s.t.

__axioms__:

(a) Compatibility

(i) $\psi_{IJ}\psi_{JK} = \psi_{IK}$, $I \subseteq J \subseteq K$

$\psi_{II} = \mathrm{id}$ $(\Rightarrow U_{II} = X_I)$

ii) $s_I \psi_{IJ} = s_I$

iii) $U_{IJ_1} \cap U_{IJ_2} = U_{I(J_1 \cup J_2)}$

and

$U_{II} = X_I$

iv) $\psi_{IJ}^{-1}(U_{IK})$

$\parallel$

$U_{JK} \cap \left(S_{J\setminus I}\big|_{X_J}\right)^{-1}(0)$

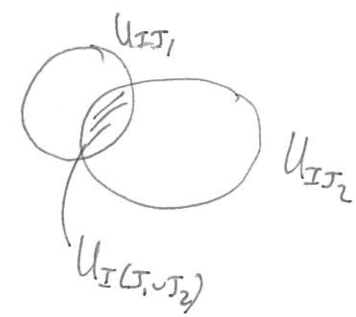
$\forall\ I \subseteq J \subseteq K$.

The ψ_{IJ} can now be __forgotten__, that's the point. The bits that are K-thickened are __equal__; ie, homeomorphic.

ii)

$$\begin{tikzcd}
\left(S_{J\setminus I}\big|_{X_J}\right)^{-1}(0) \arrow[rr, "\psi_{IJ}"] \arrow[d, phantom, sloped, "\subset"] & & U_I \arrow[d, phantom, sloped, "\subset"] \\
X_J \arrow[dr, "s_I"'] & \circlearrowleft & X_I \arrow[dl, "s_I"] \\
& E_I &
\end{tikzcd}$$

iii) Inside X_I,



v) skip; we ignore Γ

vi) $\psi_{IJ}^{-1}(X_I^{reg}) \subseteq X_J^{reg}$

thickening regular locus, you still get something regular.

b) Submersion / transversality

$$s_{J\setminus I}: X_J \longrightarrow E_{J\setminus I}$$

is locally modelled (smoothly) by

$$\mathbb{R}^{d+\dim E_I} \times \mathbb{R}^{\dim E_{J\setminus I}} \longrightarrow \mathbb{R}^{\dim E_{J\setminus I}}$$

over $\psi_{IJ}^{-1}(X_I^{reg}) \subseteq X_J^{reg}$ by (vi)

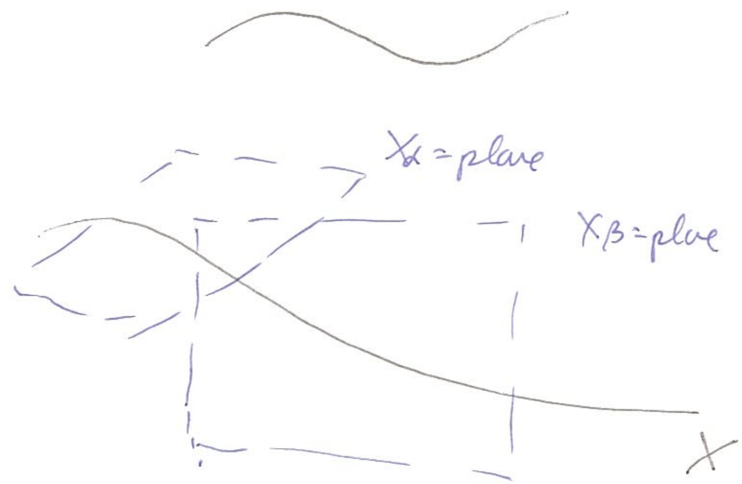
ie, thickening X_I to X_J looks like a "trivial" thickening by a smth $\mathbb{R}^{d+\dim E_I}$-mfld locally, on regular locus.

c) Covering: $\forall x \in X$, $\exists I \subset A$ finite s.t.

$$x \in \psi_{\emptyset I}\left((s_I|_{X_I^{reg}})^{-1}(0) \right).$$

Ex $X_\alpha, X_\beta, X_{\alpha\beta}$

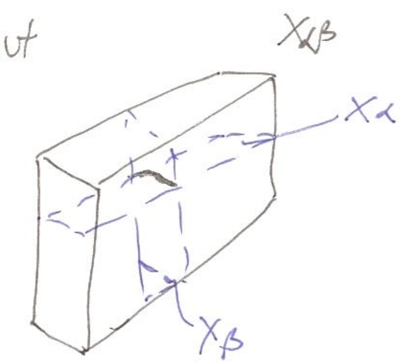
Suppose X is



on overlap,

~

put



and specify s_α so s_β cuts out X_α

" s_β " s_α cuts out X_β

from $X_{\alpha\beta}$.

Ex

$X = \mathbb{R}_x$

$X_\alpha = \mathbb{R}^2_{x,y}$

$X_\beta = \mathbb{R}^2_{x,z}$

$X_{\alpha\beta} = \mathbb{R}^3_{x,y,z}$

and define

$s_\alpha : X_{\alpha\beta} \longrightarrow \mathbb{R} = E_\alpha$

$s_\beta : X_{\alpha\beta} \longrightarrow \mathbb{R} = E_\beta$

by $s_\alpha = y$

$s_\beta = z$.

This gives imp. atlas on X where $X \cong \mathbb{R}_{std}$.

Rmk Usually, A indexes perturbation data, not open sets