DGFT Lecture 4 - Intro to Implicit Atlases

Friday, October 14, 2016 2:09 PM

(Jake McNamara speaking)

Problem:

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

(e.g.

$$\begin{tikzcd} X \cap Y \arrow[r] \arrow[d] & X \arrow[d, hook] \\ Y \arrow[r, hook] & Z \end{tikzcd}$$

e.g.:

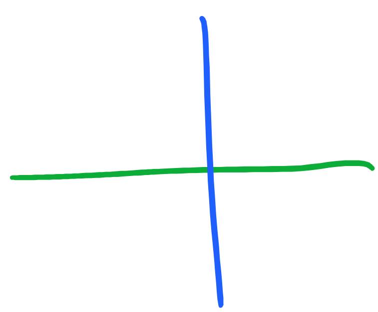
$$\begin{tikzcd} s^{-1}(0) \arrow[r] \arrow[d] & X \arrow[d, "s"] \\ * \arrow[r] & E \end{tikzcd}$$

)

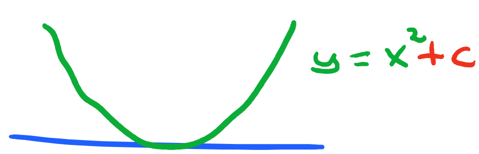
If $f \pitchfork g$, no problem. But in general, three types of problems:

$$X \times_Z Y \text{ is } \begin{cases} \text{(a) not a mfld} \\ \text{(b) the wrong dimension} \\ \text{(c) just wrong} \end{cases}$$

e.g (b):



eg (c):



$\rightsquigarrow [X \times_Z Y]$ not invt under perturbation

Spivak/Lurie: Use ideas from algebra:

$$\mathrm{Tor}^{C^\infty(Z)}\left(C^\infty(X), C^\infty(Y)\right)$$

⊛ Pardon approach:

Defn (sketch): Let X be a Hausdorff space. An implicit atlas on X is an open covering $\{U_\alpha\}$ with

"Implicit Chart" $\left\{ \forall \alpha, \ (X_\alpha, E_\alpha, s_\alpha, \psi_\alpha) \right.$

X_α: smooth mfld

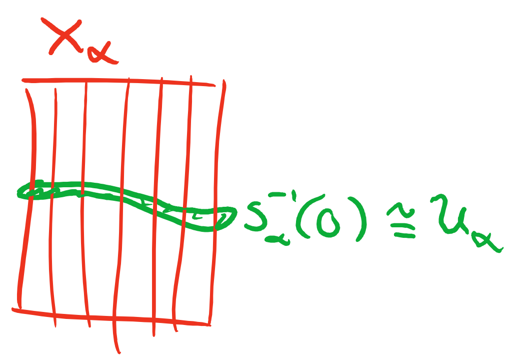
E_α: f.d. v.space

$s_\alpha : X_\alpha \to E_\alpha$

$\psi_\alpha : s_\alpha^{-1}(0) \xrightarrow{\sim} U_\alpha$

with some compatibility

Think:



Compatibility?

cuts out same U_α

$$(X_\alpha, E_\alpha, s_\alpha) \sim (X_\alpha \times \mathbb{R}, E_\alpha \times \mathbb{R}, s_\alpha \times \mathrm{id}_{\mathbb{R}})$$

- $\mathrm{vdim}(U_\alpha) = \dim(X_\alpha) - \dim(E_\alpha)$

- **Defn** [Pardon 3.1.1]:

Let X = Hausdorff space. An <u>implicit atlas</u> of vdim d on X is an index set A consists of:

DATA

(a) $\forall \alpha$, E_α = f.d. v.space

(b) For $I \subseteq A$, I finite, a Haus. space X_I with $X_\emptyset = X$

(c) $\alpha \in I \overset{\text{finite}}{\subseteq} A$,

$$s_\alpha : X_I \longrightarrow E_\alpha$$

Notation: $I \subseteq J \overset{\text{finite}}{\subseteq} A$

$$\underbrace{s_I}_{\bigoplus_{\alpha \in I} s_\alpha} : X_J \longrightarrow E_I := \bigoplus_{\alpha \in I} E_\alpha$$

(d) $I \subseteq J \overset{\text{f.}}{\subseteq} A$, a homeo^m

$$\psi_{IJ} : (s_{J \setminus I}|_{X_J})^{-1}(0) \xrightarrow{\sim} U_{IJ}$$

where

$$U_{IJ} \subseteq X_I \quad \text{open}$$

(e) Regular locus $X_I^{\text{reg}} \subseteq X_I$ open smooth mfld with s_α smooth on X_I^{reg}.

satisfying

AXIOMS

AXIOMS

(a) Compatibility

(i) $\psi_{IJ}\psi_{JK} = \psi_{IK}$

with $\psi_{II} = \mathrm{id}$

(ii) $s_I \psi_{IJ} = s_J$

$$\left(\begin{tikzcd} X_J \arrow[drr] & (s_{J\setminus I}|_{X_J})^{-1}(0) \arrow[l, hook'] \arrow[r] & U_{IJ} \arrow[r, hook] & X_I \arrow[dl] \\ & & E_I & \end{tikzcd}\right)$$

(iii) $U_{IJ_1} \cap U_{IJ_2} = U_{I,J_1 \cup J_2}$

and $U_{II} = X_I$

(iv) $\psi_{IJ}^{-1}(U_{IK}) = U_{JK} \cap (s_{J\setminus I}|_{X_J})^{-1}(0)$

(<u>Think</u>: lets you forget the ψ_{IJ} and consider intersections of $U_{IJ} \cap U_{JK}$ formally)

(v) Skipped for now (orbifold version)

(vi) $\psi_{IJ}^{-1}(X_I^{reg}) \subseteq X_J^{reg}$

(b) Submersion / Transversality

The map $S_{J\setminus I} : X_J \longrightarrow E_{J\setminus I}$

is locally modelled (smoothly) by

$$\mathbb{R}^{d+\dim E_I} \times \mathbb{R}^{\dim E_{J\setminus I}} \longrightarrow \mathbb{R}^{\dim E_{J\setminus I}}$$

$$\mathbb{R}^{d+\dim E_I} \times \mathbb{R}^{\dim E_{J\setminus I}} \longrightarrow \mathbb{R}^{\dim E_{J\setminus I}}$$

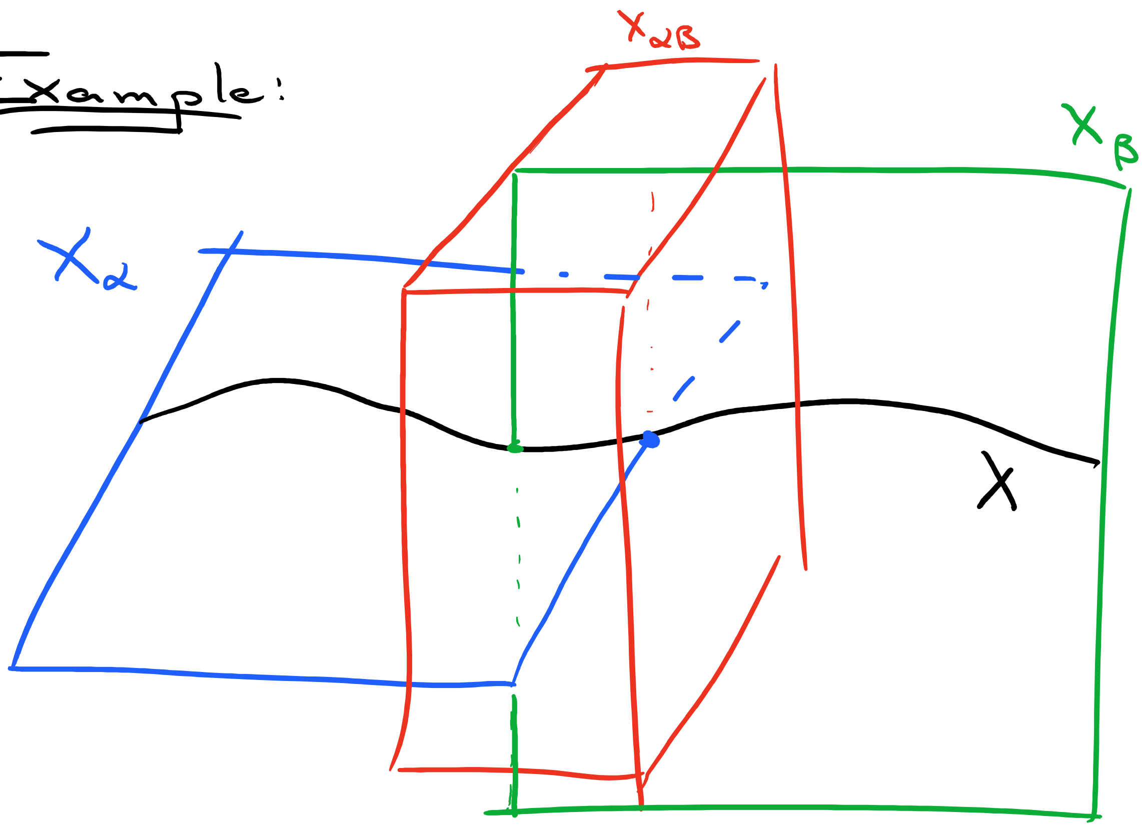
over $\psi_{IJ}^{-1}(X_I^{reg}) \subseteq X_J^{reg}$

(c) Covering

$\forall x \in X$, $\exists I$ s.t.

$$x \in \psi_{\phi I}\left(\left(s_I|_{X_I^{reg}}\right)^{-1}(0)\right)$$

Example:



$X = \mathbb{R}$, x

$X_\alpha = \mathbb{R}^2$, x, y

$X_\beta = \mathbb{R}^2$, x, z

$X_{\alpha\beta\gamma} = \mathbb{R}^3$, x, y, z

$s_\alpha = y$ $\quad E_\alpha = \mathbb{R}$

$s_\beta = z$ $\quad E_\beta = \mathbb{R}$