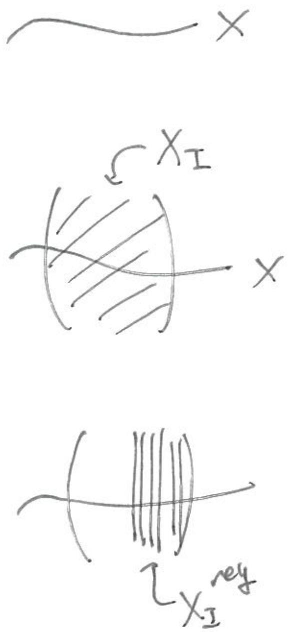
Jake

Ex.



Have:

$X = s_I^{-1}(0)$ on X_I^{reg}, s_I C^∞.

$X = s_I^{-1}(0)$ on X_I, s_I only C^0 on some Haus. top. space.

[Recap from last time + examples]

Recall: X Hausdorff
A index set

Have data of

- X_I Haus space $\forall I \subset A$ finite — "I-thickening"
- E_α fin. dim. vec. sp. $\forall \alpha \in A$ — "Obstruction space"
- Functions $s_\alpha : X_I \longrightarrow E_\alpha$ cutting out zero locus. — "Kuranishi maps"
- $U_{IJ} \subseteq X_I$ — "footprints"
- $\psi_{IJ} : (s_{J \setminus I}|_{X_J})^{-1}(0) \overset{\cong}{\underset{\text{homeo}}{\longrightarrow}} U_{IJ}$ — "footprint maps"

  (each U_{IJ} is subset of X_J cut out by $s_{J \setminus I}$.)
- $X_I^{reg} \subseteq X_I$ open. "regular locus"

Axioms:

- Compatibility (won't go over)
- Submersion

  $s_{J \setminus I} : X_J \longrightarrow E_{J \setminus I}$

  look like projection on $\psi_{IJ}^{-1}(X_I^{reg}) \subseteq X_J^{reg}$

  ie, like

  $$\begin{array}{c} \mathbb{R}^{d + \dim E_I} \times \mathbb{R}^{\dim E_{J \setminus I}} \\ \downarrow \\ \mathbb{R}^{\dim E_{J \setminus I}} \end{array}$$

  so X_J^{reg} is a top mfld.
- Covering

First example: A C^∞ mfld. Could also do C^0 case.

(In practice, showing C^∞ structure is hard; it's really about "gluing theorems" re: holom curves in practice.)

1) Let X be C^∞ mfld. Can take

$A = \{*\}$ one-point set.

$X_* := X$ trivial thickening

$E_* := 0$ 0-dim vec space

$s : X_* \longrightarrow E_*$ zero function.

$U_{\emptyset *} = X$.

$X^{reg} = X$. Nothing to do!

Note: If I took

$$X_* \longrightarrow E_* \neq 0$$

then we'd get <u>different</u> implicit mfld — the virtual dimension would be different.

<u>Rmk</u> Subm. axiom for $J = I$ means X_J^{reg} is locally $\mathbb{R}^{d+\dim E_J}$, hence a mfld.

<u>Rmk</u> The virtual dimension "d" only appears in <u>submersion</u> axiom.

<u>Ex 2</u> Now suppose X is given an (smooth) atlas

$$X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$$

Then take

$$\mathcal{A} = A$$
$$X_\alpha = U_\alpha, \quad X_I = \bigcap_{\alpha \in I} U_\alpha$$
$$E_\alpha = 0, \quad s_\alpha = 0.$$
$$U_{IJ} = X_J \subset X_I.$$

<u>Rmk</u> $\mathcal{A}$ could be used to index thickenings, or open subsets of X, or both!

<u>Ex</u> $X = *$

$$X_\alpha = S^1 \setminus \{N\}$$
$$X_\beta = S^1 \setminus \{S\}$$
$$E_\alpha = E_\beta = \mathbb{R}.$$

Using some $X_\alpha \cong \mathbb{R} \cong X_\beta$, can model 0-dim pt. This will have equivalent atlas as $\mathbb{R}^0$.

Only concerned about <u>germ</u> of each X_I.

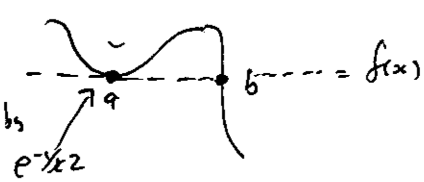
<u>Ex</u> $X_\alpha = \mathbb{R}$ $\quad$ n^{th} order pt

$E_\alpha = \mathbb{R}$

$$s_\alpha : X_\alpha \longrightarrow E_\alpha$$
$$x \longmapsto x^n$$

<u>Ex</u> $s_\alpha : X_\alpha \longrightarrow E_\alpha$

$$x \longmapsto f(x)$$



take $X_\phi^{reg} = \{b\}$

$X_\alpha^{reg} = \mathbb{R}$.

$X_\phi = \{a, b\}$

Then X models $(\mathbb{R}^0) \cup$ (germ of e^{-1/x^2} @ 0.)

Point: $\psi_{IJ}(U_{IJ}^{reg}) \hookrightarrow X_J^{reg}$ is usually a <u>strict</u> inclusion. So <u>eventually</u>, points are contained in some regular locus.

Example: Zero locus of a section of vec b

$$\underset{\text{mfld}}{Y} \xrightarrow{\ \sigma\ } F, \quad F \to Y \ C^\infty \text{ vec bdl}$$

σ section.

$\cdot X = \sigma^{-1}(0)$ $\quad$ (If F were vecspace would be done.)

Choose $\tilde{F}$ vec bdl on Y s.t.

$$\tilde{F} \oplus F \cong \underline{\mathbb{R}^N} \text{ trivial.}$$
$$\cong E \times Y.$$

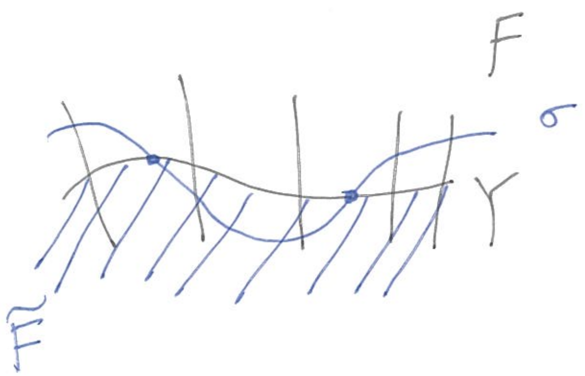
Then take $X_\phi = X$

$X_\alpha = \tilde{F}$ total space

$E_\alpha = E$ $\quad$ $s_\alpha(y, \tilde{f}) = \sigma(y) + \tilde{f}$.

$s_\alpha : \tilde{F} \longrightarrow E$

Picture:



So these E_α could be vec bdls, not just vec spaces, w/ no difference.

$\tilde{F}$ has two coords — y, and fiber $\tilde{f}$.

For s_α to equal zero, need

$$\sigma(y) \oplus \tilde{f} \in F \oplus \tilde{F}$$

to equal zero.

Here, X_ϕ^{reg} = where σ is regular,

$$X_\alpha^{reg} = X_\alpha .$$

Example $V(x^n)$.

$X_\alpha = \mathbb{R}$

$E_\alpha = \mathbb{R}$,

$s_\alpha(x) = x^n$

"order n infinitesimal thickening"

each n gives diff't implicit atlas on $\mathbb{R}^0$.

This is probably familiar from schemes, but here's a non-scheme-like example:

Example $\Omega_0 \mathbb{R}$

$X_\alpha = * = X_\phi$

$E_\alpha = \mathbb{R}$

$s_\alpha : * \mapsto 0$

virt. dim $= -1$, so a "point that shouldn't be there."

Here

$$\begin{tikzcd} \Omega_0 \mathbb{R} \arrow[r] \arrow[d] \arrow[dr, phantom, "T"] & * \arrow[d] \\ * \arrow[r] & \mathbb{R} \end{tikzcd} .$$

$E_\alpha \cong T_{x_0} X$ in deg -1.

Example

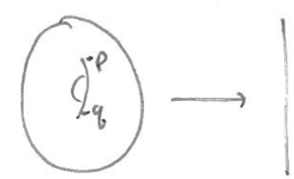
Zero section of Mob. strip, intersected w/ itself.

This is a circle w/ an infinitesimal thickening which twists once as it goes around.

This is a 0-dim implicit mfld.

What is it cobordant to? A single point. (Perturb section.)

Rmk:



non-generic Morse fcn.

Flow(p,q) has two points

so $X_\phi = * \amalg *$.

Perturbing a little, X_ϕ disappears. No flow from p to q. So we anticipate a 1-dim obstructn.

$E_\alpha = \mathbb{R}$, $X_\alpha = * \amalg *$, $s_\alpha : X_\alpha \to E_\alpha$
$* \mapsto 0$

This models "-1-dim" thickened pts

So X is a thing w/ a cob to ϕ,
which immediately becomes cobordant to ϕ.

Here, E_α is cobdr of Morse operator.