DGFT Lecture 5 - Implicit Atlases II

Friday, October 28, 2016 2:10 PM

(Dan Cristofaro-Gardiner speaking)

Example:

$$\begin{array}{c} E \\ \pi \downarrow \uparrow s \\ B \end{array}$$ with $s^{-1}(0)$ compact

A set of thickening data consisting of

$(V_\alpha, E_\alpha, \lambda_\alpha)$

(i) $V_\alpha \subseteq B$ open

(ii) E_α f.d. v-space

(iii) $\lambda_\alpha : V_\alpha \times E_\alpha \longrightarrow \pi^{-1}(V_\alpha)$

Index "set" A

$$X_I = \left\{ (x, \{e_\alpha\}_{\alpha \in I}) \;\middle|\; \begin{array}{l} x_\alpha \in \bigcap_{\alpha \in I} V_\alpha \\ \{e_\alpha\} \in \bigoplus_{\alpha \in I} E_\alpha \text{ s.t.} \end{array} \right.$$

$$\left\{ \;\middle|\; \{e_\alpha\} \in \bigoplus_{\alpha \in I} E_\alpha \text{ s.t. } \boxed{s(x) + \sum_{\alpha \in I} \lambda_\alpha(x, e_\alpha) = 0} \right\}$$

I-thickened section

Kuranishi maps

$$s_\alpha : X_I \longrightarrow E_\alpha$$

projection to E_α

Footprint

$$\mathcal{U}_{IJ} \subseteq X_I \quad \text{when } I \subseteq J$$

locus where $x \in \bigcap_{\alpha \in J} V_\alpha$

Footprint Maps

$$\psi_{IJ} : (s_{J \setminus I}|_{X_J})^{-1}(0) \longrightarrow \mathcal{U}_{IJ}$$

forgetting e_α for $\alpha \in J \setminus I$

Rmks: - This atlas is essentially canonical.

- It's huge! (Too huge?)

  $\mathcal{A}$ is not a set

- $X_I^{reg} \subsetneq X_I$

Isotropy:

Prototypical example:

Smooth orbifold

has open cover $\{X_\alpha/\Gamma_\alpha =: V_\alpha \subseteq X\}_{\alpha\in\mathcal{A}}$

where X_α smooth mflds and Γ_α = finite groups

= "covering groups"

For $I = \{\alpha_1, \ldots, \alpha_n\} \in \mathcal{A}$

$$X_I = X_{\alpha_1} \underset{X}{\times} \cdots \underset{X}{\times} X_{\alpha_n}$$

"orbifold fiber product"

$$\Gamma_I = \Gamma_{\alpha_1} \times \cdots \times \Gamma_{\alpha_n}$$

$$\Gamma_I = \Gamma_{\alpha_1} \times \cdots \times \Gamma_{\alpha_n}$$

acts smoothly on $\tilde{X}_I$

$$\tilde{X}_I / \Gamma_I = V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq X$$

This gives implicit atlas w/ isotropy where all E_α are zero.

Want to modify definition from last time to account for this:

Defn: An <u>implicit atlas of vdim d</u> on X a Hausdorff space is

Data

(0) A = index set

(i) Covering groups

$$\Gamma_\alpha = \text{finite gp}, \quad \alpha \in A$$

$$\Gamma_I = \prod_{\alpha \in I} \Gamma_\alpha, \quad |I| < \infty$$

(ii) E_α f. generated $\mathbb{R}[\Gamma_\alpha]$-modules

$$\left(E_I = \bigoplus_{\alpha \in I} E_\alpha \right)$$

(iii) Γ_I-space X_I Hausdorff,

$$X \xrightarrow{\sim} X_\emptyset$$

(iv) (Kuranishi maps)

Γ_α-equivariant

$$s_\alpha : X_I \longrightarrow E_\alpha$$

$$\forall \alpha \in I \overset{\text{fin}}{\subseteq} \mathcal{A}$$

$\left(\text{for } I \subseteq J, \quad s_I = \bigoplus_{\alpha \in I} s_\alpha : X_J \longrightarrow E_I \text{ is } \Gamma_I\text{-invt} \right)$

(v) (Footprints)

Γ_I-invt $\mathcal{U}_{IJ} \subseteq X_I$

(vi) (Footprint Maps)

Γ_J-equivariant

$$\psi_{IJ} : (s_{J \setminus I}|_{X_J})^{-1}(0) \longrightarrow \mathcal{U}_{IJ}$$

(vii) (Regular Locus)

Γ_I-invt $X_I^{reg} \subseteq X_I$

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satisfying
Axioms

(i)-(iv) Same compatibility conditions as old definition

(v) (Homeo^m axiom)

ψ_{IJ} induces homeo^m

$$(s_{J\setminus I}|_{X_J})^{-1}(0) \big/ \Gamma_{J\setminus I} \longrightarrow U_{IJ}$$

(vi) $\psi_{IJ}^{-1}(X_I^{reg}) \subseteq X_J^{reg}$

(vii) $\Gamma_{J\setminus I}$ acts freely on $\psi_{IJ}^{-1}(X_I^{reg})$

(viii) $X_I^{reg} \subseteq X_I$ open

(ix)-(x) Same submersion/covering axioms

Smooth version:

- Γ_I acts smoothly on X_I^{reg}
- s_I acts smoothly on X_I^{reg}

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- $s_{J\setminus I} : X_J \to E_{J\setminus I}$ smooth submersion over $\psi_{IJ}^{-1}(X_I^{reg})$
- ψ_{IJ} local diffeo^m over $\psi_{IJ}^{-1}(X_I^{reg})$

Also,

Version with boundary

X = compact Hausdorff

$\partial X \subseteq X$ closed

Then same data, but with Γ_I-invt closed subsets $\partial X_I \subseteq X_I$

$$\partial X_\emptyset = \partial X$$

and with

$$\psi_{IJ}^{-1}(\partial X_I) = (s_{J\setminus I}|_{\partial X_J})^{-1}(0)$$

and

and

want to allow local model

$$\mathbb{R}^{\geq 0} \times \mathbb{R}^{d + \dim E_I - 1} \times \mathbb{R}^{\dim E_{J \setminus I}} \longrightarrow \mathbb{R}^{\dim E_{J\setminus I}}$$

Nodal J-holo^c genus 0 curves with 0 marked pts in homology class β

Example:

$$\overline{\mathcal{M}}_{0,0}(X, \beta)$$

want to assume in addition when we add $r_\alpha > 2$ marked points, covering group just S_{r_α}

Basic construction similar:

Index set $(r_\alpha, D_\alpha, \lambda_\alpha, E_\alpha)$ with

(i) $r_\alpha > 2$ integer

(ii) D_α smooth codim 2 submfld w/ bdry

(iii) f.d. $\mathbb{R}[S_{r_\alpha}]$-modules E_α

(iv) $\lambda_\alpha : E_\alpha \to C^\infty(\overline{\mathcal{C}} \times X, \Omega^{0,1} \otimes TX)$

(iv) $\lambda_\alpha : E_\alpha \to C^\infty\left(\overline{\mathcal{C}}_{0,r_\alpha} \times X, \Omega^{0,1}_{\overline{\mathcal{C}}_{0,r_\alpha}/\overline{M}_{0,r_\alpha}} \otimes_{\mathbb{C}} TX\right)$

supported away from nodes and marked pts

Thickenings

- $\overline{M}_{0,0}(X,\beta)_I$ consists of
  - $u : C \longrightarrow X$ nodal, genus 0

    s.t. for $\alpha \in I$, $u \pitchfork D_\alpha$, $\# u^{-1}(D_\alpha) = r_\alpha$

    ($u^{-1}(\partial D_\alpha)$, require transversality away from nodes)
  - Elements $e_\alpha \in E_\alpha$ $\forall \alpha \in I$
  - Labellings of $u^{-1}(D_\alpha)$ by $\{1, \ldots, r_\alpha\}$
  - Satisfy Thickened $\bar\partial$ eqn

$$\bar\partial u + \sum_{\alpha \in I} \lambda_\alpha(e_\alpha)(\phi_\alpha, u) = 0$$

$\phi_\alpha : C \longrightarrow \mathcal{C}_{0,r_\alpha}$

Most axioms easy to check.

Hard part: submersion axioms near nodes