Čech cohomology and Homotopy $\mathcal{K}$-Sheaves

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0 A short intro to virtual fundamental classes

In order to define virtual fundamental classes (VFCs) over the lectures following this one, we need some results from homological algebra. This talk is essentially part of the background required to understand the VFC package. So as not to lose sight of the goal, let us start with a quick discussion of this package, and where we are headed.

The idea, at its most fundamental level (forgetting about orientation sheaves and relative versions), is the following. Associated to an implicit atlas (X, A), we will associate, over a ground ring R (we'll usually take $\mathbb{Z}$), a chain complex, $C^\bullet_{\mathrm{vir}}(X, A)$, which comes with natural maps

$$C^\bullet_{\mathrm{vir}}(X, A) \to C_{d+\dim E_I-\bullet}(E_I, E_I \setminus 0) = R[-d]$$

for any finite I. On homology, this map will not depend on the choice of I. In the case of $A = \{\alpha\}$, this map is literally just the pushforward of the Kuranishi map

$$(s_\alpha)_*\colon C^\bullet_{\mathrm{vir}}(X, \{\alpha\}) := C_{d+\dim E_\alpha-\bullet}(X^{\mathrm{reg}}_\alpha, X^{\mathrm{reg}}_\alpha \setminus X) \to C_{d+\dim E_\alpha-\bullet}(E_\alpha, E_\alpha \setminus 0).$$

Our algebraic machinery will then let us apply some sort of Poincaré duality so that the virtual cochain complex is quasi-isomorphic to Čech cohomology, and hence we obtain a map

$$\check{H}^\bullet(X) \to R[-d],$$

from which it follows that the virtual fundamental class lies in some sort of dual of Čech cohomology.

As a quick justification for why we might want to use Čech (co)homology as opposed to singular (co)homology, we consider the case of the Warsaw circle sitting in $\mathbb{R}^2$, carved out by the zero set of a function s which is negative on the interior and positive on the exterior. This is an implicit atlas of vdim 1, and we see that we expect a perturbation of s to still have zero set which winds around the annulus once, so the VFC should be nonzero. However, if the VFC were to lie in singular homology, then it would have to be zero! On the other hand, Čech homology is non-trivial. There will also be a brief comment on further justification for the choice of Čech (co)homology later.

The main key, then, is some sort of Poincaré-Lefschetz duality. We want that $C^\bullet_{\text{vir}}(X, A)$ is quasi-isomorphic to $\check{C}^\bullet(X)$ (twisted by the orientation sheaf). The virtual chain complex, in turn, will be shown to have the structure of a so-called pure homotopy $\mathcal{K}$-sheaf.

A final piece of notation - any space in what follows should be assumed to be a locally Hausdorff space. We shall simply say *space* from now on.

1 Čech cohomology for sheaves

Consider some space X, together with some open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of X. One can then form a simplicial set, called the nerve of the covering, $\mathcal{N}(\mathcal{U})$. Recall that a simplicial set consists of a collection of $\mathbb{Z}_{\geq 0}$-indexed sets together with face and degeneracy maps between them satisfying certain properties. For $k \geq 0$, we therefore define

$$\mathcal{N}(\mathcal{U})_k := \coprod_{\alpha_0, \ldots, \alpha_k \in A} (U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k})$$

The face maps are given by the possible inclusions from when an index is dropped, and the degeneracy maps are the isomorphisms from when an index is doubled. Suppressing the degeneracy maps, and using $U_{\alpha_0 \cdots \alpha_k} := U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_k}$, our simplicial set is:

$$\cdots \coprod U_{\alpha_0 \alpha_1 \alpha_2} \substack{\to \\ \to \\ \to} \coprod U_{\alpha_0 \alpha_1} \rightrightarrows \coprod U_{\alpha_0}$$

Heuristically speaking, one expects that as we take finer and finer refinements, the nerve essentially encodes all of the homotopy-theoretic information of X. In fact, if $\mathcal{U}$ is a "good" open cover, meaning all intersections of U_α's are either empty or contractible, then the geometric realization $|\mathcal{N}(\mathcal{U})|$ recovers X up to homotopy equivalence, provided that X is either paracompact or $\mathcal{U}$ is locally finite.

Recall now the following definitions:

Definition 1.1. A *presheaf* on a space X is a contravariant functor from the category of open sets of X (with inclusion morphisms) to the category of abelian groups (or any abelian category, if we'd prefer to think of R-modules instead of $\mathbb{Z}$-modules, for example). For an open subset $U \subseteq X$, and a presheaf $\mathcal{F}$ on X, elements of the group $\mathcal{F}(U)$ are called *sections*. Morphisms of presheaves are natural transformations.

Definition 1.2. A *sheaf* on a space X is a presheaf such that for any collection $\{U_\alpha\}_{\alpha \in A}$ of open subsets of X, the diagram

$$0 \to \mathcal{F}\left(\cup_\alpha U_\alpha\right) \to \prod_{\alpha \in A} \mathcal{F}(U_\alpha) \to \prod_{\alpha, \beta \in A} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact, where the last arrow is the difference between the two restrictions.

To draw out the meaning of exactness in the above diagram a little more clearly, a sheaf satisfies the conditions that:

- If there are sections $s_\alpha \in \mathcal{F}(U_\alpha)$ such that the restrictions of s_α and s_β to $U_{\alpha\beta}$ are equal for each α, β, then there is some global section $s \in \mathcal{F}(\cup_\alpha U_\alpha)$.
- If the restriction of a section $s \in \mathcal{F}(\cup_\alpha U_\alpha)$ to each $\mathcal{F}(U_\alpha)$ is zero, then s is zero. (This is equivalent to requiring uniqueness of the global section s in the previous bullet point.)

To any presheaf $\mathcal{F}$, we can associate a Čech cohomology as follows. As a first step, for a fixed open cover $\mathcal{U}$, apply $\mathcal{F}$ to the Čech nerve considered with its face maps. This yields a diagram

$$0 \to \prod \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_{\alpha\beta}) \overset{\to}{\underset{\to}{\to}} \prod \mathcal{F}(U_{\alpha\beta\gamma}) \cdots$$

Taking the alternating sum of the face maps. This yields a single cochain complex

$$0 \to \prod \mathcal{F}(U_\alpha) \to \prod \mathcal{F}(U_{\alpha\beta}) \to \prod \mathcal{F}(U_{\alpha\beta\gamma}) \to \cdots$$

Definition 1.3. The *Čech cohomology of $\mathcal{U}$ with values in $\mathcal{F}$*, notated $\check{H}^\bullet(\mathcal{U}, \mathcal{F})$, is the cohomology of the above cochain complex.

If $\mathcal{V}$ is a refinement of $\mathcal{U}$, then there is a natural map $\check{H}(\mathcal{U}, \mathcal{F}) \to \check{H}(\mathcal{V}, \mathcal{F})$. This leads to the following definition:

Definition 1.4. The *Čech cohomology* of X is

$$\check{H}^\bullet(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{H}(\mathcal{U}, \mathcal{F}).$$

Remark. Note that for any choice of open cover, the zeroth Čech cohomology is just the space of global sections, and that refinement yields isomorphisms on this group. Therefore,

$$\check{H}^0(X, \mathcal{F}) = \mathcal{F}(X).$$

Let us now provide some brief justification as to why we might care about Čech cohomology in our foray into VFCs.

- As alluded to above, the nerve of $\mathcal{U}$ is a pretty good combinatorial model for X, and is the most natural way of encoding the topology of X as a simplicial set. Our definition of Čech cohomology makes it clear that our cohomology really only depends on how $\mathcal{F}$ depends on this homotopical model for X.

- There is a purely derived geometric approach to defining a cohomology for a sheaf. In particular, the functor from sheaves to abelian groups given by taking global sections is left exact, and there are enough injective sheaves to define the right derived functors of this global section functor, which are the sheaf cohomology functors $H^i(X, \bullet)$. Leray's theorem states that if we choose our open cover $\mathcal{U}$ such that sheaf cohomology on a given sheaf $\mathcal{F}$ vanishes on all finite intersections, then the Čech cohomology $\check{H}(\mathcal{U}, \mathcal{F})$ is naturally isomorphic to its sheaf cohomology $H(X, \mathcal{F})$. In general, there is a Mayer-Vietoris type spectral sequence relating the two, and also a natural map from Čech cohomology to sheaf cohomology.

2 Homotopy $\mathcal{K}$-sheaves

A $\mathcal{K}$-(pre)sheaves

Just as presheaves are defined with respect to open sets, one can instead attempt to work with compact sets. This motivates the introduction of the $\mathcal{K}$-presheaf, and in analogy with the construction of sheaves, we shall also define $\mathcal{K}$-sheaves.

Definition 2.1. A *$\mathcal{K}$-presheaf* on a space X is a contravariant functor from the category of compact sets of X (with inclusion morphisms) to the category of abelian groups. Morphisms of $\mathcal{K}$-presheaves are natural transformations.

Definition 2.2. A $\mathcal{K}$-*sheaf* $\mathcal{F}$ on a space X is a $\mathcal{K}$-presheaf such that

- $\mathcal{F}(\emptyset) = 0$
- For any two compact subsets K_1, K_2 of X, we obtain an exact sequence

$$0 \to \mathcal{F}(K_1 \cup K_2) \to \mathcal{F}(K_1) \oplus \mathcal{F}(K_2) \to \mathcal{F}(K_1 \cap K_2)$$

- For any compact subset $K \subset X$, the following natural map is an isomorphism:

$$\varinjlim_{\substack{K \subseteq \overline{U} \\ U \text{ open}}} \mathcal{F}(\overline{U}) \to \mathcal{F}(K)$$

Remark. In this definition, and for the rest of this talk, it is important that we are taking X to be locally compact Hausdorff, so just recall that this condition is lying inside the word 'space'.

It turns out that there is a natural adjunction between presheaves and $\mathcal{K}$-presheaves which is an equivalence on the restriction to sheaves and $\mathcal{K}$-sheaves. Hence, we can consider these two to essentially be the same.

We note that in the compact world, stalks are easier to talk about. We shall just write $\mathcal{F}_p := \mathcal{F}(\{p\})$.

B Digression on total complexes

We now have our category of ($\mathcal{K}$)-(pre)sheaves as a basic object. The derived geometry approach would suggest that we might want to consider the derived category, and so we should consider complexes of these objects, and study mapping cones, and so on. For this, we introduce the following notation. Suppose

$$A_0^\bullet \xrightarrow{f_0} \cdots \xrightarrow{f_{n-1}} A_n^\bullet$$

is a complex of complexes (e.g. of ($\mathcal{K}$)-(pre)sheaves). Then we define

$$[A_0^\bullet \to \cdots \to A_n^{\bullet - n}]$$

to be the associated total complex of this double complex. That is, if E is this complex, then we take

$$E_k = \bigoplus_{i+j=k} A_i^j$$

with differential $\partial\colon E_k \to E_{k+1}$ given by its restriction to each A_i^{k-i} as

$$\partial|_{A_i^{k-i}} = \partial_{A_i} + (-1)^k f_i.$$

The fact that this forms a chain complex is equivalent to the f_i's being chain maps composing to zero.

For example, suppose we consider a morphism of complexes $f\colon A^\bullet \to B^\bullet$. Then the complex $C(f) := [A^\bullet \to B^{\bullet-1}]$ is called the mapping cone, with differential

$$\partial_{C(f)} = \begin{pmatrix} \partial_A & 0 \\ (-1)^k f & \partial_B \end{pmatrix}.$$

In this example, we obtain a natural exact sequence of cohomology groups

$$H^\bullet(C(f)) \to H^\bullet(A) \xrightarrow{f_*} H^\bullet(B) \xrightarrow{+1}$$

coming from the long exact sequence in cohomology from the short exact sequence of chain complexes

$$0 \to B^{\bullet-1} \to C(f)^\bullet \to A^\bullet \to 0.$$

I leave it as an exercise to show that the connecting morphism induced from this SES really is f_*.

Generalizing a little further, we have the following example. Consider the complex of $\mathcal{K}$-presheaves $\mathcal{F}^\bullet(K) := C_{n-\bullet}(M, M \setminus K)$. Then one can consider, for pairs of compact subsets $K_1, K_2 \subseteq X$, the complex

$$[\mathcal{F}^\bullet(K_1 \cup K_2) \to \mathcal{F}^{\bullet-1}(K_1) \oplus \mathcal{F}^{\bullet-1}(K_2) \to \mathcal{F}^{\bullet-2}(K_1 \cap K_2)].$$

The point is that acyclicity of this complex is a rephrasing of the Mayer-Vietoris long exact sequence. In particular, we have that the total complex is just the iterated mapping cone

$$\left[\mathcal{F}^\bullet(K_1 \cup K_2) \to \left[\mathcal{F}^\bullet(K_1) \oplus \mathcal{F}^\bullet(K_2) \to \mathcal{F}^{\bullet-1}(K_1 \cap K_2)\right]^{\bullet-1}\right]$$

and so acyclicity of this total complex is equivalent to

$$H^\bullet\mathcal{F}(K_1 \cup K_2) \simeq H^\bullet\left(\left[\mathcal{F}^\bullet(K_1) \oplus \mathcal{F}^\bullet(K_2) \to \mathcal{F}^{\bullet-1}(K_1 \cap K_2)\right]\right),$$

and the right hand side, as a cone, itself fits into an LES from the previous example. This yields the Mayer-Vietoris exact sequence as desired.

One final note for this section, as a last chance to get used to notation, and so that we can phrase results in a more easily generalized manner. We can rephrase our definition of Čech cohomology a little more compactly using this total complex notation. Namely,

$$\check{H}(X,\mathcal{F}) = \varinjlim_{X=\cup_{\alpha\in A}U_\alpha} H^\bullet\left[\bigoplus_{p\geq 0}\prod_{S\subseteq A|S|=p+1}\mathcal{F}\left(\bigcap_{\alpha\in S}U_\alpha\right)[-p]\right],$$

where the morphisms in the complex is given by the Čech differential as defined above (the alternating sum of the face maps).

This motivates the following:

Definition 2.3. We define the *Čech cohomology of a complex $\mathcal{F}^\bullet$ of $\mathcal{K}$-presheaves on a compact space X* by:

$$\check{H}(X,\mathcal{F}^\bullet) := \varinjlim_{X=\cup_{i=1}^n K_i} H^\bullet\left[\bigoplus_{p\geq 0}\bigoplus_{1\leq i_0<\cdots<i_p\leq n}\mathcal{F}^{\bullet-p}\left(\bigcap_{j=0}^{p}(K_{i_j})\right)\right].$$

We note that the limit is taken over finite open covers.

This definition matches with usual Čech cohomology when $\mathcal{F}^\bullet$ is just a $\mathcal{K}$-sheaf (concentrated in degree zero) under the natural equivalence between sheaves and $\mathcal{K}$-sheaves. (This is Lemma A.4.11 in Pardon's paper.)

C Homotopy $\mathcal{K}$-sheaves

Definition 2.4. A *homotopy $\mathcal{K}$-sheaf* is a complex of $\mathcal{K}$-presheaves $\mathcal{F}^\bullet$ such that

- $\mathcal{F}^\bullet(\emptyset)$ is acyclic
- (Mayer-Vietoris) For pairs of compact subsets $K_1, K_2 \subseteq X$, the complex

  $$\left[\mathcal{F}^\bullet(K_1\cup K_2)\to\mathcal{F}^{\bullet-1}(K_1)\oplus\mathcal{F}^{\bullet-1}(K_2)\to\mathcal{F}^{\bullet-2}(K_1\cap K_2)\right]$$

  is acyclic.
- For any compact subset $K\subset X$, the following natural map is a quasi-isomorphism:

  $$\varinjlim_{\substack{K\subseteq\overline{U}\\ U\text{ open}}}\mathcal{F}^\bullet(\overline{U})\to\mathcal{F}^\bullet(K)$$

Remark. It is expected that the category of homotopy $\mathcal{K}$-sheaves is equivalent to the category of so-called homotopy sheaves (defined analogously), or at least when we consider bounded complexes, but it ends up being easier just to work with these $\mathcal{K}$-sheaves directly instead of cooking up such an equivalence.

Remark. A $\mathcal{K}$-sheaf, thought of as a complex of $\mathcal{K}$-presheaves concentrated in degree zero, is not necessarily a homotopy $\mathcal{K}$-sheaf. For this, we need surjectivity of the restriction maps, called *softness*, and a complex of soft $\mathcal{K}$-sheaves is automatically a homotopy $\mathcal{K}$-sheaf.

We will want to check that our virtual chain complexes are actually homotopy $\mathcal{K}$-sheaves. At first, we will only define 'partial' virtual chain complexes, and then take a homotopy colimit to obtain our full virtual chain complex. For the 'partial' complexes, consider the following two lemmas.

Lemma 2.5 (Pardon Lemma A.6.3)**.** *The complex of $\mathcal{K}$-presheaves given by*

$$\mathcal{F}^\bullet(K) = C_{-\bullet}(X, X \setminus K)$$

is a homotopy $\mathcal{K}$-sheaf.

Lemma 2.6 (Pardon Lemma A.2.11)**.** *If $\mathcal{F}^\bullet$ has a finite filtration with associated graded a homotopy $\mathcal{K}$-sheaf, then $\mathcal{F}^\bullet$ is a homotopy $\mathcal{K}$-sheaf.*

These two lemmas will immediately tell us that our so-called 'partial' virtual chain complexes will be homotopy $\mathcal{K}$-sheaves. The boundary version will just be this singular homology presheaf (shifted in degree), and the full version will look like a mapping cone of such pieces. We might worry about gluing these partial complexes together in a later talk.

We finish this section by observing that a homotopy $\mathcal{K}$-sheaf computes its own $\check{H}^\bullet$:

Proposition 2.7 (Pardon Proposition A.4.14)**.** *If $\mathcal{F}^\bullet$ is a homotopy $\mathcal{K}$-sheaf (in fact, if it satisfies only the first two conditions), then the natural map*

$$H^\bullet\mathcal{F}^\bullet(X) \to \check{H}^\bullet(X, \mathcal{F}^\bullet)$$

is an isomorphism.

D Pure homotopy $\mathcal{K}$-sheaves

Actually, being a homotopy $\mathcal{K}$-sheaf isn't strong enough. We present the following stronger condition, which our virtual chain complexes will satisfy.

Definition 2.8 (Pardon Definition A.5.1)**.** A homotopy $\mathcal{K}$-sheaf is said to be *pure* when

- (Stalk cohomology) $H^i\mathcal{F}^\bullet_p = 0$ for $i \neq 0$.
- (Weak vanishing) $H^i\mathcal{F}^\bullet = 0$ locally for $i << 0$ (meaning for all p, there is a neighborhood U with $H^i\mathcal{F}^\bullet(K) = 0$ for all compact $K \subset U$).

The key property is a refinement of homotopy $\mathcal{K}$-sheaves computing their own homology, which works for pure homotopy $\mathcal{K}$-sheaves.

Proposition 2.9 (Pardon Proposition A.5.4)**.** *For $\mathcal{F}^\bullet$ a pure homotopy $\mathcal{K}$-sheaf, there is a canonical isomorphism*

$$H^\bullet\mathcal{F}^\bullet(X) = \check{H}^\bullet(X, H^0\mathcal{F}^\bullet).$$

Further, for a complex of $\mathcal{K}$-presheaves

$$\mathcal{F}^\bullet_0 \to \cdots \to \mathcal{F}^\bullet_n$$

with each $\mathcal{F}_i$ a pure homotopy $\mathcal{K}$-sheaf, we obtain a canonical isomorphism

$$H^\bullet\left[\mathcal{F}^\bullet_0(X) \to \cdots \to \mathcal{F}^{\bullet-n}_n(X)\right] = \check{H}(X, \left[H^0\mathcal{F}^\bullet_0 \to \cdots \to H^0\mathcal{F}^\bullet_n[-n]\right])$$

Remark. For a pure homotopy $\mathcal{K}$-sheaf, it is automatic that $H^0\mathcal{F}^\bullet$ is a $\mathcal{K}$-sheaf.

Proof sketch. I am including this just to show that there's a lot being swept under the rug here. We break this into three isomorphisms:

First, we have the map

$$H^\bullet\left[\mathcal{F}^\bullet_0(X) \to \cdots \to \mathcal{F}^{\bullet-n}_n(X)\right] \to \check{H}^\bullet(X, \left[\mathcal{F}^\bullet_0(X) \to \cdots \to \mathcal{F}^{\bullet-n}_n(X)\right]),$$

which is an isomorphism because $[\mathcal{F}^\bullet_0(X) \to \cdots \to \mathcal{F}^{\bullet-n}_n(X)]$ is a homotopy $\mathcal{K}$-sheaf, which computes its own $\check{H}^\bullet$.

Second, the map

$$\check{H}^\bullet(X, \left[\mathcal{F}^\bullet_0(X) \to \cdots \to \mathcal{F}^{\bullet-n}_n(X)\right]) \to \check{H}^\bullet(X, \left[\tau_{\geq 0}\mathcal{F}^\bullet_0(X) \to \cdots \to \tau_{\geq n}\mathcal{F}^{\bullet-n}_n(X)\right])$$

is an isomorphism, where $\tau_{\geq i}$ restricts to the positive degree part, because it turns out that pure homotopy sheaves actually satisfy strong vanishing $H^i\mathcal{F}^\bullet = 0$ for $i < 0$, and $\check{H}^\bullet$ preserves quasi-isomorphisms. These are two lemmas that we glossed over.

Finally, it turns out that a map of homotopy $\mathcal{K}$-sheaves, each of which satisfies a vanishing condition a la strong vanishing described above, which is a quasi-isomorphism on stalks will induce an isomorphism on $\check{H}$. So we obtain an isomorphism

$$\check{H}(X, \left[H^0\mathcal{F}_0^\bullet \to \cdots \to H^0\mathcal{F}_n^\bullet[-n]\right]) \to \check{H}^\bullet(X, \left[\tau_{\geq 0}\mathcal{F}_0^\bullet(X) \to \cdots \to \tau_{\geq n}\mathcal{F}_n^{\bullet-n}(X)\right]).$$

This another whole batch of lemmas leading to a proposition that we have simply glossed over. □

E Poincaré duality

Purity of our partial virtual chain complexes will come down to an explicit computation via Poincaré Duality. This is the goal towards which we work at the end of the section and the end of the talk. We will not present a full version for simplicity.

First a technical note. In order to define Poincaré duality, we actually need to use compactly supported Čech cohomology on a sheaf instead. To define this is a two step process:

- For fixed compact K, $\check{H}^\bullet_K(X, \mathcal{F})$ is defined exactly the same as $\check{H}^\bullet(X, \mathcal{F})$, but with sections $\mathcal{F}(U)$ replaced with $\ker \mathcal{F}(U) \to \mathcal{F}(U \setminus K)$.
- Then take $\check{H}^\bullet_c(X, \mathcal{F}) := \varinjlim_{K \subseteq X} \check{H}^\bullet_K(X, \mathcal{F})$.

For an inclusion of an open set $f\colon X \hookrightarrow Y$, and a sheaf $\mathcal{F}$ on X, one can define a new sheaf, $f_!\mathcal{F}$, on Y. Namely, $(f_!\mathcal{F})(U)$ consists of elements of $\mathcal{F}(f^{-1}(U))$ which vanish in a neighborhood of $Y \setminus X$.

Now any map f (not just an inclusion) pulls back open covers, and so for example, with f an inclusion, we obtain a map

$$f_!\colon \check{H}^\bullet_c(X, \mathcal{F}) \to \check{H}^\bullet_c(Y, f_!\mathcal{F}).$$

Lemma 2.10 (Pardon A.4.7)**.** *This map is an isomorphism.*

Now let's specialize.

Definition 2.11. On a topological manifold M, the *orientation sheaf* is the sheaf

$$\mathfrak{o}_M(U) := H_{\dim M}(M, M \setminus U).$$

For a topological manifold with boundary, let $j\colon M \setminus \partial M \to M$ denote inclusion. Then define

$$\mathfrak{o}_M := j_* \mathfrak{o}_{M \setminus \partial M}$$

$$\mathfrak{o}_{M \text{ rel } \partial} = j_! \mathfrak{o}_{M \setminus \partial M}$$

Finally, we have a simple version of the Poincaré duality we need. (Pardon proves a relative version, which is no harder to prove, but moderately more annoying to state.) For an inclusion $i\colon A \to B$, we can define a pullback map of $\mathcal{K}$-sheaves given by $(i^*\mathcal{F})(K) = \mathcal{F}(i(K))$. This allows us to state the result:

Lemma 2.12 (Pardon Lemma A.6.4, simple version)**.** *Let M a topological n-manifold with boundary. Let $i\colon X \hookrightarrow M$ the inclusion of a closed subset. Then*

$$H_{n-\bullet}(M, M \setminus X) = \check{H}_c(X, i^* \mathfrak{o}_{M \text{ rel } \partial}).$$

Proof Sketch. We have $C_{n-\bullet}(M, M \setminus X)$ is a homotopy $\mathcal{K}$-sheaf. We can check that it is pure because it is easy to check that the stalks are all $\mathbb{Z}$ in degree zero at points in the interior of M, and otherwise zero. If we take the 1-point compactification $f\colon X \to X^+$, we see that we have an isomorphism of stalks $f_! i^* \mathfrak{o}_M \to H^0 \mathcal{F}^\bullet$ inducing an isomorphism of sheaves. This therefore yields, from our computation of homology of pure homotopy $\mathcal{K}$-sheaves, that:

$$H^\bullet \mathcal{F}^\bullet(X^+) = \check{H}^\bullet(X^+, f_! i^* \mathfrak{o}_M),$$

and the right-hand side is just $\check{H}_c(X, i^* \mathfrak{o}_M)$ from our previous lemma. □

The previous version covers one of our partial virtual cochains. The following more general version covers the other.

Lemma 2.13 (Pardon Lemma A.6.4, full version)**.** *Consider also $N \subseteq \partial M$ a tamely embedded codim 0 submanifold with boundary, and let j be the inclusion* $\operatorname{Int} M \cup \operatorname{Int} N \hookrightarrow M$*. Then*

$$H^\bullet \left[C_{n-1-\bullet}(N, N \setminus X) \to C_{n-\bullet}(M, M \setminus X) \right] = \check{H}^\bullet_c(X, i^* j_! j^* \mathfrak{o}_M)$$