Ryan] VFCs

Pt I: Gluing:

idea: Have a large implicit atlas

$X_\alpha, E_\alpha, s_\alpha, \psi_{ij}$, etc.

On each subset of index set

$J \subset A$

given vdim $X = d$, want to produce fund. cycles

$[U_\alpha]_\alpha^{vir} : \check{H}^d(U_\alpha) \longrightarrow \mathbb{Z} \quad \forall \alpha$

easy for each piece; the question is how to glue to produce

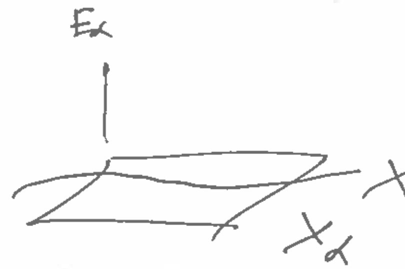
$$[X]_A^{vir} : \check{H}^d(X) \longrightarrow \mathbb{Z}.$$

Assume simplest thing: X has single chart.

$$\begin{array}{c} E_\alpha \\ \uparrow s_\alpha \\ X_\alpha \end{array}, \quad X = s_\alpha^{-1}(0).$$

Then easily by Poinc. duality:

$$\check{H}^d \cong H_{\dim X_\alpha - d}(X_\alpha, X_\alpha \setminus X)$$



pushforward:

$$\check{H}^d \cong H_{\dim X_\alpha - d}(X_\alpha, X_\alpha \setminus X) \xrightarrow{(s_\alpha)_*} H_{\dim X_\alpha - d}(E_\alpha, E_\alpha \setminus 0)$$

So all the composite

$$\check{H}^d(X) \xrightarrow{[X]_\alpha^{vir}} \mathbb{Z}$$

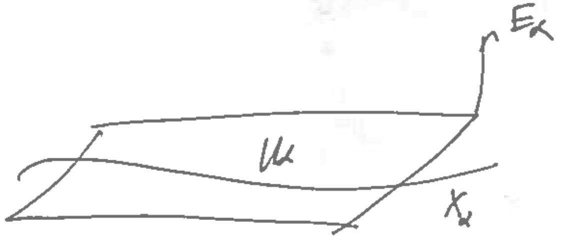
the map we seek.

Need to glue together using

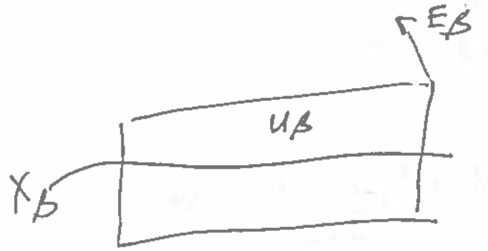
$$\psi : (s^{-1}_{J \setminus I}|_{X_J}(0)) \longrightarrow U_{I, IJ}$$

Today: Two charts

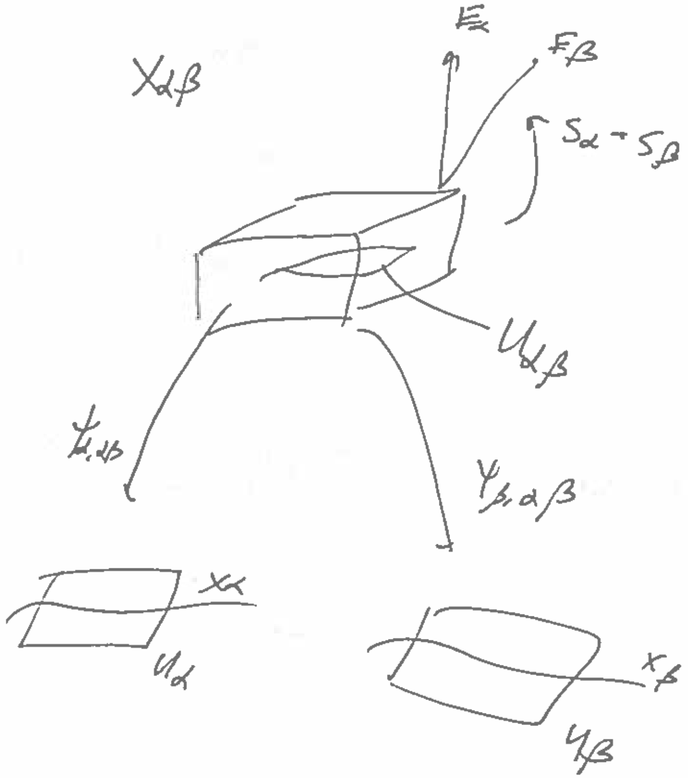
So now ...



$U_f = U_{\phi\alpha}$



and a space



$X_{\alpha\beta} \neq X_\alpha \times X_\beta$, more bizarre

First try:

Consider mapping cone by "intersection of gluings":

$$C_{\dim X_\alpha - *}(X_\alpha, X_\alpha | U_\alpha)$$

chain rel

glue via:

$$\begin{tikzcd} & C_{\dim X_\alpha - *}(X_\alpha, X_\alpha | U_\alpha) \\ C_{\dim X_{\alpha\beta} - *}(X_{\alpha\beta}, X_{\alpha\beta} | U_{\alpha\beta}) \arrow[ur, "*"] \arrow[dr, "*"'] & \\ & C_{\dim X_\beta - *}(X_\beta, X_\beta | U_\beta) \end{tikzcd}$$

$*$ should be like "cap" w/ $[S_\beta^{-1} T_0]$

$[S_\alpha^{-1} T_0]$.

③

Capping loses dimension, so fix that first.

Roughly, what is $\psi_{\alpha,\alpha\beta}$?

Looks like

$$X_\alpha \times E_\beta \overset{(\ast\ast)}{\longleftarrow\!\sim\!\sim} X_{\alpha\beta}$$

linearizing $\qquad\qquad$ curved

where $\dim(X_\alpha \times E_\beta) = \dim(X_{\alpha\beta})$.

But we don't have a map like $(\ast\ast)$

So what to do?

$$C_{\dim X_\alpha + \dim E_\beta - \ast}\left(X_\alpha \times E_\beta,\ X_\alpha \times E_\beta \setminus U_\alpha \times E_\beta\right)$$

$$C_{\dim X_{\alpha\beta} - \ast}(X_{\alpha\beta}, X_{\alpha\beta} \setminus U_{\alpha\beta})$$

dimensions now fixed.

What's map?

Have

$$\psi_{\beta,\alpha\beta} : s_\beta^{-1}\big|_{X_{\alpha\beta}}(0) \longrightarrow U_{\beta,\alpha\beta}$$

fixpoint

but don't have maps like

$$U_{\beta,\alpha\beta} \times E_\alpha \longleftarrow\!-\!-\!\to X_{\alpha\beta} \longleftarrow\!-\!-\!\to U_{\alpha,\alpha\beta} \times E_\beta$$

Fix: Deform to normal cone.

"Linearizing fixpoint @ a pt":

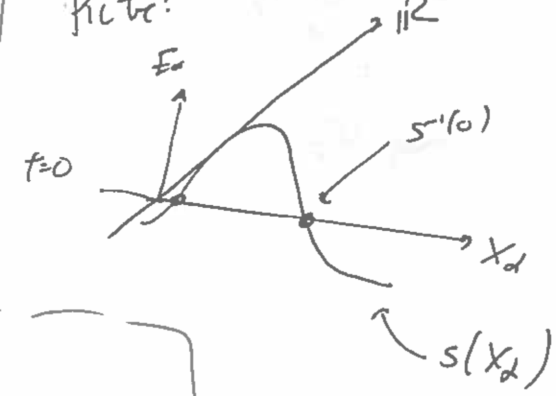
Fix

$$\mathbb{R}_{\geq 0} \times E_\alpha \times X_{\alpha\beta}$$

$$\cup$$

$$\{(t, e_\alpha, x) \text{ s.t. } s_\alpha(x) = t e_\alpha\}$$

Picture:



away from $t=0$, just have a copy of s $\forall t$.

As $t \to 0$, get zero set,

plus E_α could be anything.

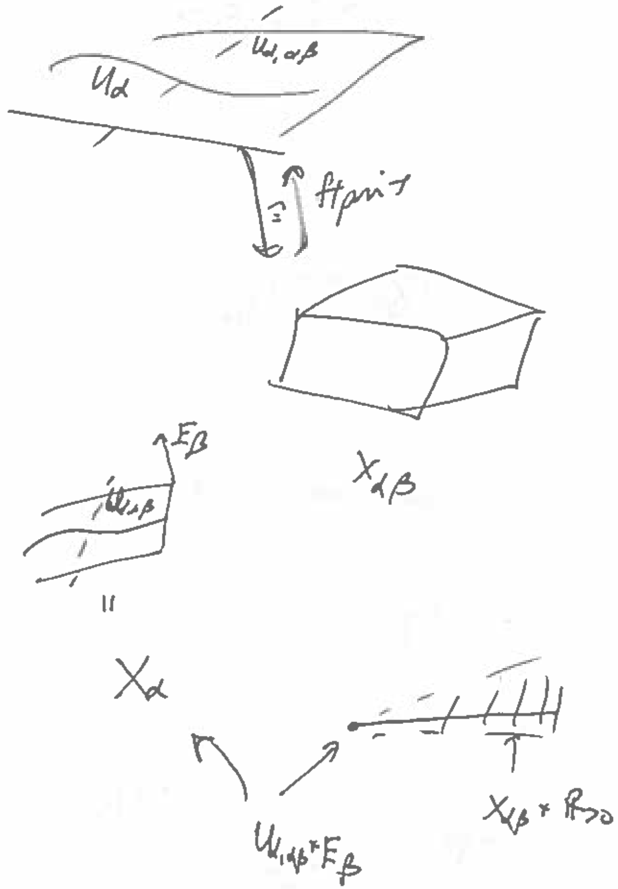
$$S_\alpha^{-1}(0) \times E_\alpha \quad @\ t=0$$

copy of $X_{\alpha\beta}$ @ $t > 0$.

So get maps:

• by footprint, have

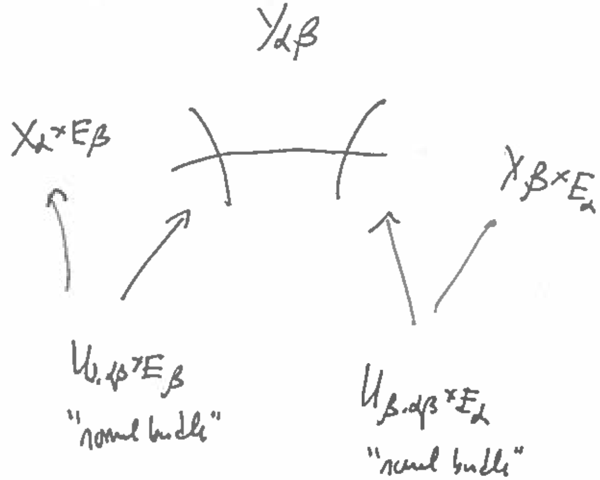
$$\begin{tikzcd} & S_\alpha^{-1}(0)\times E_\alpha \arrow[dl, shift left, "\psi_{\alpha,\alpha\beta}"'] \\ U_{\alpha,\alpha\beta}\times E_\alpha \arrow[ur, shift right, "s^{-1}"'] & \end{tikzcd}$$



Then, at $t=1$, put in a copy of s_β, too:

$$\mathbb{R}_{\geq 0}\times E_\alpha\times E_\beta\times X_{\alpha\beta} \supset Y_{\alpha\beta}$$

$$= \left\{ \begin{array}{l} s_\alpha(z) = t e_\alpha \\ s_\beta(z) = (1-t)e_\beta. \end{array} \right\}$$



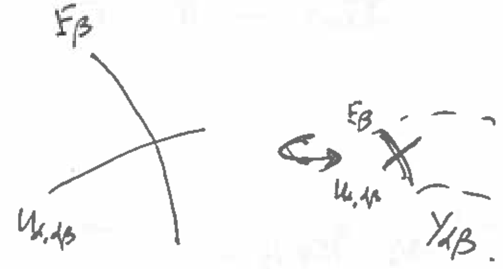
Upshot: gluing 3 spaces via 2 subspaces

$$\begin{tikzcd}[column sep=small]
X_\alpha \times E_\beta & & Y_{\alpha\beta} & & X_\beta \times E_\alpha \\
& U_{\alpha,\alpha\beta} \times E_\beta \arrow[ul] \arrow[ur] & & U_{\beta,\alpha\beta} \times E_\alpha \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

E_α define type of deformations you can make.

$Y_{\alpha\beta}$, · like relative mapping cylinder.

also, · <u>universal</u> for objects allowing for · deformation of $U_{\alpha,\alpha\beta}$ in E_β direction.



mapping cone via chains:

$$\begin{tikzcd}[column sep=tiny]
& C_{\dim X_{\alpha\beta} - \bullet}(X_\alpha \times E_\beta,\ X_\alpha \times E_\beta \setminus U_\alpha \times 0) \\
C_{\dim X_{\alpha\beta} - \bullet}(U_{\alpha,\alpha\beta} \times E_\beta,\ U_{\alpha,\alpha\beta} \times E_\beta \setminus U_\alpha \times 0) \arrow[ur] \arrow[dr] & \\
& C_{\dim X_{\alpha\beta} - \bullet}(Y_{\alpha\beta},\ Y_{\alpha\beta} \setminus 0 \times [0,1] \times U_{\alpha\beta}) \\
C_{\dim X_{\alpha\beta} - \bullet}(U_{\beta,\alpha\beta} \times E_\alpha,\ U_{\beta,\alpha\beta} \times E_\alpha \setminus U_\beta \times 0) \arrow[ur] \arrow[dr] & \\
& C_{\dim X_{\alpha\beta} - \bullet}(X_\beta \times E_\alpha,\ X_\beta \times E_\alpha \setminus U_\beta \times 0)
\end{tikzcd}$$

every space here has natural map to $E_\alpha \times E_\beta$!

So has a map s_* to

$$C_{\dim X_{\alpha\beta} - \bullet}(E_\alpha \oplus E_\beta\ ,\ E_\alpha \times E_\beta \setminus 0) \cong \mathbb{Z}[-d].$$

<u>H:</u> Everything in category of spaces <u>over</u> $(E_\alpha \times E_\beta,\ E_\alpha \times E_\beta \setminus 0)$.

Defn

$$X_{I,J,A} \subset \mathbb{R}^A \times E_A \times X_J^{reg}$$

$$\| $$

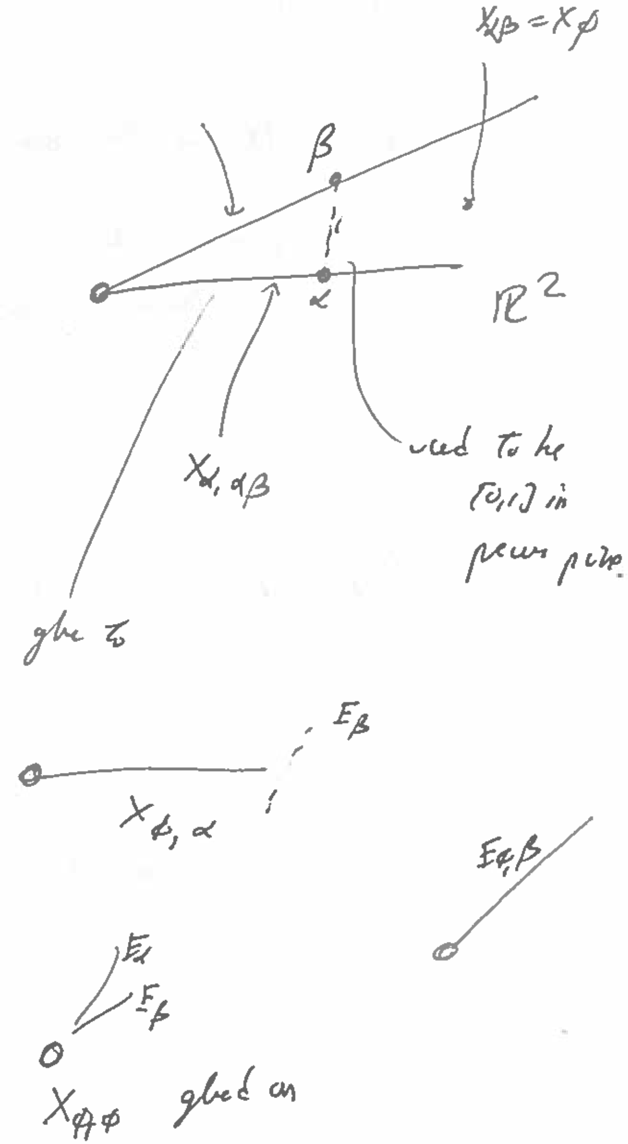
$$\{ (\{t_\alpha\}, \{e_\alpha\}, x) \text{ st}$$

- $t_\alpha = 0 \quad \alpha \in A \setminus I$
- $s_\alpha(x) = t_\alpha e_\alpha \quad \alpha \in J$
- avoid transversality issues:

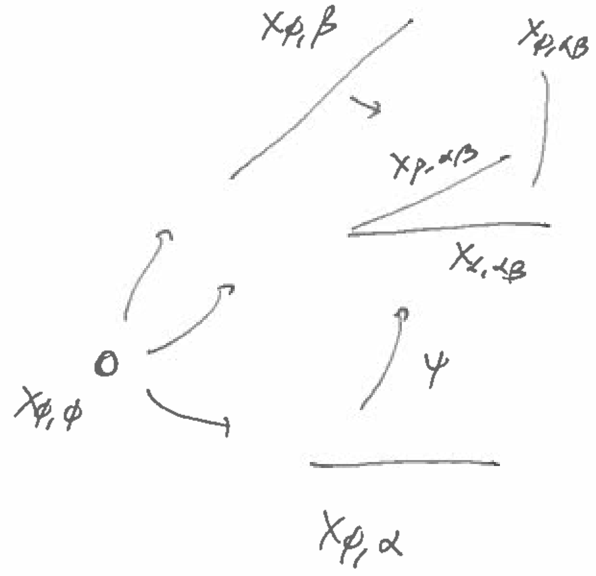
$$\psi_{\{\alpha \in A,\, t_\alpha > 0\},\, J}(x) = X^{reg}_{\{\alpha \in A,\, t_\alpha > 0\}}$$

"don't blow up regular pnts"

Ex $A = \{\alpha, \beta\}$



glue:



if pt not reg in $X_{\phi,\alpha}$, ψ pushes it to $X_{\alpha,\alpha\beta}$ where it's maybe going to become regular.

Given $K \subset X$ compact,

$$X^K_{I,J,A} = \{x \in \psi^{-1}_{\phi J}(K)\}$$

now use: htpy K sheaves:

$$C^\bullet_{vir}(K, A)_{IJ} = C_{d+\dim E_A - \bullet}(X_{I,J,A}, X_{I,J,A} \setminus X^K_{I,J,A})$$

Remarks: Can prove this is

1) homotopy K sheaf

2) pure, etc

Then

$$C^\bullet_{vir}(K,A) := \operatorname{hocolim}_{I \subseteq J \subseteq A} C^\bullet_{vir}(K,A)_{IJ}.$$

$$\downarrow$$

$$C_{d+\dim E_A}(E_A, E_A \setminus 0)$$

So $[X]^{vir}_A$ is defined to be

$$[X]^{vir}_A : H^\bullet_{vir}(X) \longrightarrow H_{d+\dim E_A - \bullet}(E_A, E_A \setminus 0) \cong \mathbb{Z}[-d].$$

Thm:

$$H^\bullet_{vir}(X) \cong \check{H}^\bullet(X).$$